Exact renormalization group and applications to disordered problems: part I

February 1, 2008

Pascal Chauve[1]
Laboratoire de Physique des Solides
Université Paris-Sud, Bât. 510
91405 Orsay - France

Pierre Le Doussal[2]
Laboratoire de Physique Théorique
École Normale Supérieure
24, rue Lhomond
75231 Paris Cédex 05 - France

Abstract

We develop a systematic multi-local expansion of the Polchinski-Wilson exact renormalization group (ERG) equation. Integrating out explicitly the non local interactions, we reduce the ERG equation obeyed by the full interaction functional to a flow equation for a function, its local part. This is done perturbatively around fixed points, but exactly to any given order in the local part. It is thus controlled, at variance with projection methods, e.g. derivative expansions or local potential approximations. Our method is well-suited to problems such as the pinning of disordered elastic systems, previously described via functional renormalization group (FRG) approach based on a hard cutoff scheme. Since it involves arbitrary cutoff functions, we explicitly verify universality to $\mathcal{O}(\epsilon = 4 - D)$, both of the $T = 0$ FRG equation and of correlations. Extension to finite temperature T yields the finite size (L) susceptibility fluctuations characterizing mesoscopic behaviour $\overline{(\Delta\chi)^2} \sim L^\theta/T$, where θ is the energy exponent. Finally, we obtain the universal scaling function to $\mathcal{O}(\epsilon^{1/3})$ which describes the ground state of a domain wall in a random field confined by a field gradient, compare with exact results and variational method. Explicit two loop exact RG equations are derived and the application to the FRG problem is sketched.

[1]E-mail: chauve@lpt.ens.fr
[2]E-mail: ledou@lpt.ens.fr

1 Introduction

The idea of writing an exact equation for the scale dependence of the full action functional already appears in the review of Wilson and Kogut [1]. Since it is an equation for a full functional of the fields, its detailed analysis is hindered by technical complications. The much simpler Wilson momentum shell [1] integration method is commonly used for one loop calculations. Since it does not follow the full functional, subsequent efforts were made to embed it into a better controlled sharp cutoff exact RG [2, 3]. For practical perturbative calculations beyond one loop, field theoretical renormalization methods are more often used since they have proved vastly more efficient. However, the exact RG equations offer the hope to develop ab initio calculation relying on no assumption, possibly non perturbative, from any bare model. In principle it should be useful to obtain precise results when applied to bare theories for which we have little insight on possible underlying field theoretical description.

In the work of Polchinski [4], the exact RG equation was put on a more precise and aesthetic framework, and used to prove the renormalizability of the ϕ^4 theory in four dimensions. The exact renormalization group equations indeed provide formal results or general proofs about symmetries [5]. For practical calculations however, one needs to truncate in some way these highly complicated functional equations. To do so, different procedures have been proposed [6, 7], and have been mainly applied to the study of non-perturbative problems [8]. For example the exponents of the $O(n)$ model in three dimensions were estimated [9] using a choice of truncation. One commonly used projection method is the so called local potential approximation [10], obtained by a constant background field method neglecting the momentum dependence. Further extensions include additional projections on higher gradients of the field [11]. Although very interesting, these projection methods are often uncontrolled. More accurate results are expected if more couplings are kept, which is possible with heavy numerical integrations of flow equations. In this respect, exact RG as a tool is now used both in particle and condensed matter physics. For instance, outstanding problems in strongly correlated electrons such as the Hubbard model in $D = 2$, have been recently studied by numerically integrating the flow of a large number of vertices[12], using a fermionic version of the Polchinski equation[4].

By contrast, comparatively a few works use exact RG method to develop *perturbative* calculations. One example is the computation of the beta function of ϕ^4 in $4-\epsilon$ dimensions to one loop[13], where universality is made particularly explicit through the use of an arbitrary cutoff function. Although obviously more powerful methods are available in that case, there are some problems in condensed matter physics which appear within reach of perturbative calculations but for which no coherent field theoretical formulation is available at present. This is the case for the pinning of an elastic system in a random potential, for which a momentum shell RG method has been developed[14]. In this problem, an infinite number of coupling constants becomes relevant for $D < D_c = 4$ and one must write a RG equation for a full function $R(u)$ (the second cumulant of the disorder), hence the name "functional" renormalization group (FRG). As such it differs from standard field theoretical RG. Thus, to understand better this problem, i.e to show explicitly universality to one loop and beyond, there is a need for a perturbatively controlled exact RG method, able to admit a full function, the local part $R(u)$ as a small parameter. Indeed the field theoretical formulation is frought with difficulties, in particular because the function $R(u)$ develops non analytic behaviour at

finite scale. These issues are discussed in a related work [15].

In this paper we develop a novel method to solve the Polchinski exact renormalization group equation and use it for explicit calculations. Writing the action as a sum of multilocal interactions, we note that the Polchinski equation naturally reduces to a hierarchy of equations obeyed by simple functions. This hierarchy can be solved in an expansion in powers of the local part. Indeed, we find that exact integration of the multilocal parts yields a single RG equation for the local part. The method is thus controlled around fixed points where the local part is proportional to a well defined small parameter (e.g. $\epsilon = 4 - D$). It does not require any arbitrary projection procedure or neglect of operators, as is usually done in derivative expansions or local potential approximations. In addition, we obtain explicit formulas for any correlation function which allow for practical calculations. Since this is done for arbitrary cutoff functions, it allows explicit check of universality order by order in the expansion.

The aim of this paper is twofold. On the one hand we present the general method to all orders, valid for a large class of theories. We derive the explicit form of the exact RG equation for the local interaction up to third order. On the other hand we apply this method to several problems, first as a check, to the $O(n)$ model, and second to the FRG for disordered elastic systems. Explicit calculations and applications in this paper are restricted mainly to one loop. Although briefly mentionned here, applications to two loops will be detailed in a companion paper [16].

Two variants of the method are presented. The most direct one consists in a straight expansion of the action in multilocal terms. The second one consists instead of first absorbing tadpoles into the interaction (so-called Wick ordering), then expanding. Being inequivalent, they provide independent checks of the universal results. The first method yields more complicated equations but, can be better suited to some problems, such as the $T > 0$ FRG. Note that although Wick ordered versions of the Polchinski equation have been studied before, the multilocal expansion performed here is to our knowledge novel.

As mentionned above, the method is indeed well-suited to the FRG for disordered elastic systems of internal dimension D since there the full local part is controlled by $\epsilon = 4 - D$. It allows us to show that the one loop FRG equation, as well as correlation functions, are independent of the cutoff function. In addition we obtain higher cumulants of the renormalized disorder, which as the second cumulant, are non analytic functions. This is necessary to escape the so-called dimensional reduction [18], i.e the property of the present theory by which all perturbative calculations at $T = 0$ are identical to the same calculation in a trivial gaussian theory [17] (see Appendix D). This nonanalytic behaviour is rounded at finite temperature T and we obtain the scaling form of the rounding region. This allows us to compute, for the first time using the FRG method, the susceptibility fluctuations which characterize the glassy behaviour of finite size systems. Finally, we obtain the universal $\mathcal{O}(\epsilon^{1/3})$ correlation function which describes the ground state of a domain wall in a random field confined by a field gradient, compare with exact results and variational method.

The method presented in this paper also allows to investigate the theory of disordered elastic systems beyond lowest order in ϵ (one loop). A recent two loop calculation was presented in [44]. However since it was performed at $T = 0$, and for analytic $R(u)$ it fails beyond a finite length (Larkin length) and cannot describe universal properties. The application of exact FRG to next order is described in [16]. We sketch here however some

preliminary results.

The paper is organized as follows. In Section 2.1 we present in a pedagogical way the conventional exact RG method. Appendix A and Appendix B provide complements, respectively about general invariance properties of the correlations and about examples of solvable cases of the Polchinski equation. In 2.2, the multilocal expansion in the local part U is introduced up to bilocal terms. The ensuing RG equations to order U^2 are given in 2.3. The multilocal expansion to arbitrary order and the RG equation to order U^3 is given in Appendix C. The multilocal expansion of the Wick ordered functional up to second order, and the resulting one loop RG equation is presented in 3.1. The general multilocal expansion (Appendix C.1) and the resulting form to third order is given in Appendix C.2. The explicit two loop RG equation is obtained in C.3. Application to the $O(n)$ model to one loop is presented in 3.2. We then turn to applications to disordered elastic systems in Section 4 (one loop) and Section 5 (two loops). First we recall and generalize in Appendix D the dimensional reduction phenomenon. Then the $T = 0$ FRG equations are established in Section 4.1 and finite temperature extension are given in Section 4.2. Finally, the calculations of the scaling function in the random field Ising model is performed in 4.3. We sketch in Appendix E the variational calculation to be compared with the FRG results of 4.3 and sketch some preliminary steps of a two loop FRG in Appendix F.

2 Method

2.1 Exact RG procedure

Consider a system whose state is described by a bosonic *field* $\phi_i^x = \phi_i(x)$, where x denotes position in space, and i is a general label denoting e.g. fields indices, spin, replica indices, additional coordinate (e.g. time) etc. (or more generally any quantity which will not undergo the coarse-graining). The system, in the presence of external *sources* J_i^x, is described by the partition function:

$$Z(J) = \int_\phi e^{J:\phi - \mathcal{S}(\phi)} \tag{1}$$

obtained by the integration over the field ϕ, where the *action* $\mathcal{S}(\phi)$ is a functional of the field ϕ, and $J:\phi$ denotes here and in the following the full scalar product (e.g. $\int_x \sum_i J_i^x \phi_i^x$, with $\int_x \equiv \int d^D x$). In a problem of equilibrium statistical mechanics, $\mathcal{S}(\phi) = \mathcal{H}(\phi)/T$ where $\mathcal{H}(\phi)$ is the hamiltonian and T the thermodynamic temperature, the free energy being $F = -T \ln Z(0)$. Averages of any *observable* $\mathcal{A}(\phi)$ (i.e. functional of ϕ) are defined by

$$\langle \mathcal{A}(\phi) \rangle_{\mathcal{S}} = \frac{\int_\phi \mathcal{A}(\phi) e^{-\mathcal{S}(\phi)}}{\int_\phi e^{-\mathcal{S}(\phi)}} \tag{2}$$

The usual way to compute correlation functions and averages is to perform a perturbation expansion, writing the action as a sum of a quadratic part and a non-linear part $\mathcal{V}(\phi)$

$$\mathcal{S}(\phi) = \frac{1}{2}\phi : G^{-1} : \phi + \mathcal{V}(\phi) \tag{3}$$

where $\mathcal{V}(\phi)$ a functional of ϕ and $G_{ij}^{xy} = G_{ji}^{yx}$ is a symmetric invertible matrix, $\phi : G^{-1} : \phi = \sum_{ij} \int_{xy} \phi_i^x (G^{-1})_{ij}^{xy} \phi_j^y$. In the following we denote the Gaussian average of any observable by $[\mathcal{A}(\phi)]_G = \langle \mathcal{A}(\phi) \rangle_{\mathcal{S}_G}$ with respect to the quadratic theory $\mathcal{S}_G = \frac{1}{2}\phi : G^{-1} : \phi$. We introduce the generating function of all correlation functions

$$W(J) = \ln \left[e^{J:\phi - \mathcal{V}(\phi)} \right]_G \tag{4}$$

Note that it differs from the usual definition by a J-independent quantity $\ln Z(J) = W(J) + \frac{1}{2}\mathrm{Tr} \ln G$. The ultra-violet cutoff, present in physical models, is necessary to yield finite results in the perturbative calculation with respect to $\mathcal{V}$. A broad class of soft cutoffs can be implemented on the Gaussian part, giving a vanishing weight to fast fields. For example a scalar massive theory, rotationnally invariant, is regularized in the UV by the following general cutoff function

$$G(q) = \frac{c(\frac{q^2}{2\Lambda^2})}{q^2 + m^2} \tag{5}$$

where $c(0) = 1$ and $c(s)$ decreases rapidly to zero for $s > 1$ as in Figure 1.

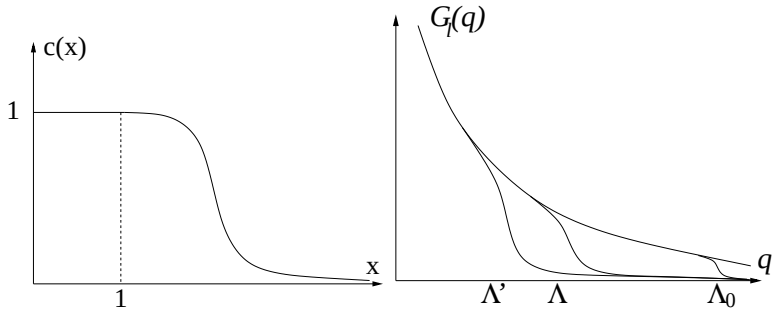

Figure 1: Cutoff function.

The exact RG method [1, 4] consists in varying the cutoff Λ and writing an equation for the function $\mathcal{V}(\phi)$ so as to conserve exactly the averages of all observables involving only "slow" modes of the field. More precisely, the average of an observable $\mathcal{A}(\phi)$ depending only on modes $q < \Lambda'$ of the field ϕ can be computed within any of the theories linked by the equation presented by Polchinski corresponding to a cutoff $\Lambda > \Lambda'$. As in any RG procedure, the strategy will be to compute averages of slow observables using the coarse-grained theory of cutoff $\Lambda' \ll \Lambda_0$.

To this aim, a set of actions $\mathcal{S}_l$

$$\mathcal{S}_l(\phi) = \frac{1}{2}\phi : G_l^{-1} : \phi + \mathcal{V}_l(\phi) \tag{6}$$

is introduced, where the Gaussian part is an arbitrary function G_l of l (e.g. corresponding to a cutoff $\Lambda_l = \Lambda_0 e^{-l}$). The initial propagator, corresponding to a cutoff Λ_0, is denoted $G_{l=0} \equiv G$ and the bare interaction $\mathcal{V}_{l=0} \equiv \mathcal{V}$. The correlation functions in $\mathcal{S}_l$ derive from $W_l(J) = \ln \left[e^{J:\phi - \mathcal{V}_l(\phi)} \right]_{G_l} = \ln Z_l(J) - \frac{1}{2}\mathrm{Tr} \ln G_l$.

For any given Λ', one defines a (Λ'–)slow observable to be a functional of ϕ depending only on the $\phi^q = \phi(q)$ with $q < \Lambda' = \Lambda_0 e^{-l'}$. We want to choose the l-dependent non-linear part

$\mathcal{V}_l(\phi)$ so that the averages of slow observables remain unchanged. Through differentiation, it is equivalent to ensure that

$$\partial_l W_l(J) \text{ independent of } J \tag{7}$$

for *any* source J with $J^q = 0$ for $q > \Lambda'$. Using the general identity

$$\partial_l[\mathcal{A}_l(\phi)]_{G_l} = \frac{1}{2}\mathrm{Tr}\left(\partial_l G_l : [\frac{\delta^2}{\delta\phi\delta\phi}\mathcal{A}_l(\phi)]_{G_l}\right) + [\partial_l \mathcal{A}_l(\phi)]_{G_l} \tag{8}$$

valid for Gaussian averages of any l-dependent observable $\mathcal{A}_l(\phi)$ and applying it to $\mathcal{A}_l(\phi) = e^{J:\phi - \mathcal{V}_l(\phi)}$, one finds

$$\partial_l[e^{J:\phi-\mathcal{V}_l(\phi)}]_{G_l} = \left[\left(\frac{1}{2}(J - \frac{\delta}{\delta\phi}\mathcal{V}_l(\phi)) : \partial_l G_l : (J - \frac{\delta}{\delta\phi}\mathcal{V}_l(\phi)) - \partial_l \mathcal{V}_l(\phi) - \frac{1}{2}\mathrm{Tr}\partial_l G_l : \frac{\delta^2}{\delta\phi\delta\phi}\mathcal{V}(\phi)\right) e^{J:\phi-\mathcal{V}_l(\phi)}\right]_{G_l}$$

where here and in (8), $\mathrm{Tr}A : B \equiv \sum_{ij}\int_{xy} A_{ij}^{xy} B_{ji}^{yx}$. Hence, if $\mathcal{V}_l(\phi)$ satisfies the Polchinski functional equation

$$\partial_l \mathcal{V}_l(\phi) = -\frac{1}{2}\mathrm{Tr}\left(\partial_l G_l : \frac{\delta^2}{\delta\phi\delta\phi}\mathcal{V}_l(\phi)\right) + \frac{1}{2}\frac{\delta}{\delta\phi}\mathcal{V}_l(\phi) : \partial_l G_l : \frac{\delta}{\delta\phi}\mathcal{V}_l(\phi) \tag{9}$$

then the above conservation condition (7) is satisfied. We have used explicitely the condition

$$J : \partial_l G_l = 0 \tag{10}$$

which imposes that the cutoff function verifies $\partial_l G_l^q = 0$ for $q < \Lambda'$ and $l > l'$. Hence, for the example (5) one has to choose [19] cutoff functions $c(s)$ such that $c(s) = 1$ for $0 \leq s \leq s_0$ with some (arbitrary) s_0.

The above framework is in fact too restrictive. We can easily lift the restriction on slow modes (and on the form of the cutoff function $c(s)$). The applications of Polchinski equation can be generalized to the computation of *any* observable (not restricted to be "slow"). As shown in Appendix A.1 one can indeed express $W(J)$ in terms of any of the l-dependent actions $\mathcal{S}_l(\phi)$:

$$W(J) = \frac{1}{2}J : (G - G : G_l^{-1} : G) : J + W_l(J : G : G_l^{-1}) \tag{11}$$

In fact we show in Appendix A.2 an even more general method which allows for arbitrary field rescalings.

Differentiating $W(J)$ once yields $\langle\phi\rangle_{\mathcal{S}} = G : G_l^{-1}\langle\phi\rangle_{\mathcal{S}_l}$, and once again yields the two point connected correlation function

$$\langle\phi\phi\rangle_{\mathcal{S}}^c = G + G : G_l^{-1} : \left(\langle\phi\phi\rangle_{\mathcal{S}_l}^c - G_l\right) : G_l^{-1} : G \tag{12}$$

and so on for higher correlations. When performing a perturbative calculation, the factors $G : G_l^{-1}$ restore the original propagator for the external lines, whereas internal lines of the

graph involve G_l. Accordingly, with this procedure the function $c(s)$ can be arbitrary (it is however convenient - see below - to use $c'(0) = 0$).

Note that if $G : J = G_l : J$, one recovers (7), i.e. $W(J) = W_l(J)$ for these slow J's as a special case of (5). In that case, for $q < \Lambda'$, (12) reduces to $\langle\phi\phi\rangle^c_{\mathcal{S}} = \langle\phi\phi\rangle^c_{\mathcal{S}_l}$ as it should.

To compute correlation functions, it is useful to express $W(J)$ in a perturbation expansion in powers of $\mathcal{V}_l(\phi)$, which reads to lowest order

$$W(J) = \frac{1}{2}J : G : J - e^{-\frac{1}{2}J:G:G_l^{-1}:G:J}\left[e^{J:G:G_l^{-1}:\phi}\mathcal{V}_l(\phi)\right]_{G_l} + \mathcal{O}(\mathcal{V}_l^2) \tag{13}$$

The Polchinski equation (9) can equivalently be written as a functional "diffusion" equation

$$\partial_l e^{-\mathcal{V}_l(\phi)} = -\frac{1}{2}\mathrm{Tr}\partial_l G_l : \frac{\delta^2}{\delta\phi\delta\phi}e^{-\mathcal{V}_l(\phi)} \tag{14}$$

or in its integrated form

$$e^{-\mathcal{V}_l(\phi)} = \left[e^{-\mathcal{V}_0(\phi+\phi')}\right]_{G-G_l} \tag{15}$$

where the average is over ϕ', which makes explicit the definition of $\mathcal{V}_l(\phi)$ as a coarse-grained interaction, i.e. integrated over the "fast part" ϕ' of the field. In fact, the decomposition into slow and fast modes and the definition of coarse-grained observables relies on the property $[\mathcal{A}(\phi)]_G = \left[[\mathcal{A}(\phi+\phi')]_{G-G_l}\right]_{G_l}$ of Gaussian averages (see Appendix A.1).

Although in general the Polchinski equation is far too complicated to be solved, in some simple cases one can find exact solutions e.g. Gaussian models, zero dimensional toy model. Most interestingly, there exist a large class of exact solutions which appear as superpositions of gaussians. In all these cases, one can explicitly verify an interesting property of the Polchinski equation to generate cusp singularities. This is further discussed in Appendix B and in forthcoming publication [45].

2.2 Multi-local expansion

The Polchinski equation, in addition to being elegant, is conceptually more satisfactory than other RG methods, e.g. Wilson's shell renormalization, because it is exact and better controlled since. Being valid for arbitrary cutoff procedures, it does not suffer from the problems associated with the sharp cutoff [3]. However, this functional equation generates non-local operators, which until now, has limited its practical applications. This generation can be seen in terms of Feynman diagrams and compared to Wilson's shell renormalization, since $G - G_l$ which contains a range of wave-vectors centered around $\Lambda_0 e^{-l}$, plays the role of the on-shell propagator. The term $\mathcal{V}''$ with a second derivative in (9) represents tadpoles while the term $\mathcal{V}'\mathcal{V}'$ represents diagrams with only one contraction (one particle reducible). These last terms are non-local operators. For instance in ϕ^4 theory, it generates the operator $\phi(x)^3\partial G^{x-y}\phi(y)^3$ which is bi-local since it corresponds to a graph where external momenta must be greater than Λ_l. The way Polchinski's equation reproduces the loop diagrams (i.e. local terms) is that after integration over a slice dl, a bilocal interaction generated by the second term of (9) is fed into tadpole diagrams. A fast momentum goes around the

corresponding loop, and slow external momenta are allowed. Thus one needs to integrate the flow and study the feedback of the generated non-local operators into local ones.

We now present a method which allows to perform this program in a controlled way. The following expansion in the number of points (local, bi-local, etc..)

$$\mathcal{V}(\phi) = \int_x U(\phi^x) + \int_{xy} V(\phi^x, \phi^y, x-y) + \dots \tag{16}$$

is valid a priori for any translationally invariant functional $\mathcal{V}(\phi)$ interactions. We discuss here only the first two terms, the general systematics being given in Appendix C. Here, $U(\phi)$ is a function of the vector ϕ_i and involves the value of the field at one point in space. The bi-local part is a function $V(\phi, \psi, z)$ of two vectors ϕ, ψ and a space coordinate difference z. In order that the expansion be well-defined, one needs the bi-local interactions to have *no* projection on the local ones. A natural way to define such a projection, inspired from the conventional short-distance-expansion, is the exact equality

$$\begin{aligned} \int_{xy} F(\phi^x, \phi^y, x-y) &= \int_{xy} F(\phi^x, \phi^x, y) + \int_{xy} \Big(F(\phi^{x+y/2}, \phi^{x-y/2}, y) - F(\phi^x, \phi^x, y) \Big) & (17) \\ &= \int_x (\overline{P}_1 F)(\phi^x) + \int_{xy} ((1-P_1)F)(\phi^x, \phi^y, x-y) & (18) \end{aligned}$$

where we have introduced the projections

$$\begin{aligned} (\overline{P}_1 F)(\phi) &= \int_y F(\phi, \phi, y) & (19) \\ (P_1 F)(\phi, \psi, z) &= \delta(z) \int_y F(\phi, \psi, y) & (20) \end{aligned}$$

on the subspaces of local and bi-local interactions respectively. Indeed, $(\overline{P}_1(1-P_1)F)(\phi) = \int_x ((1-P_1)F)(\phi, \phi, x) = 0$, i.e. $(1-P_1)F$ has zero local part and is thus properly bi-local. Interestingly, this definition implies that the function $V(\phi, \psi, z)$ appearing in the proper bi-local operator of (16) also satisfy the stronger property $\int_z V(\phi, \psi, z) = 0$ for any ϕ, ψ. Note that with no loss of generality, $V(\phi, \psi, z) = V(\psi, \phi, -z)$. Here in addition, we will consider parity invariant theories ($V(\phi, \psi, z) = V(\phi, \psi, -z)$ too).

For theories where the initial interaction U is local and is formally treated as a "small" quantity U (e.g. the ϕ^4 theory in $D = 4 - \epsilon$ where $U \sim \epsilon$), it is natural to consider that the bi-local term will be of higher order $\mathcal{O}(U^2)$. In fact, this property results from the Polchinski equation since the term which creates bilocal interactions from local ones is $\mathcal{O}(U^2)$ (the first part $\mathcal{V}''$ does not increase the degree of non-locality of $\mathcal{V}$). This property that solutions of the Polchinski equation can be organized in powers of U depending on their locality holds to arbitrary orders (p-local operators are $\mathcal{O}(U^p)$) as is discussed below and shown in Appendix C. Thus, to lowest non-trivial order $\mathcal{O}(U^2)$, the flow equations involve local and bilocal parts. Their schematic structure is

$$\begin{aligned} \partial U &= U'' + \overline{P}_1 (V'' + U'U') + \mathcal{O}(UV) + \mathcal{O}(V^2) & (21) \\ \partial V &= (1-P_1)(V'' + U'U') + \mathcal{O}(UV) + \mathcal{O}(V^2) & (22) \end{aligned}$$

where we have written the subdominant terms which will be neglected in the following.

A simplification occurs if we choose, as done in this Section, $(\partial_l G_l)_{ij}(q=0)=0$. Indeed, the term $\overline{P}_1 U' U'$ vanishes since

$$\int_x (\partial_l G_l^x)_{ij} \partial_i U(\phi) \partial_j U(\phi) = 0 \tag{23}$$

Let us write now (21,22) in an explicit form:

$$\partial_l U_l(\phi) = -\frac{1}{2} \partial G_{ij}^{x=0} \partial_i \partial_j U_l(\phi) - \int_x \partial G_{ij}^x \partial_i^1 \partial_j^2 V_l(\phi,\phi,x) \tag{24}$$

$$\partial_l V_l(\phi,\psi,x) = -\partial_i^1 \partial_j^2 \left(\partial G_{ij}^x V_l(\phi,\psi,x) - \delta(x) \int_y \partial G_{ij}^y V_l(\phi,\psi,y) \right) \tag{25}$$

$$-\frac{1}{2} \partial G_{ij}^{x=0} \left(\partial_i^1 \partial_j^1 + \partial_i^2 \partial_j^2 \right) V_l(\phi,\psi,x) + \frac{1}{2} \partial G_{ij}^x \partial_i U_l(\phi) \partial_j U_l(\psi) \tag{26}$$

where ∂G stands for $\partial_l G_l$ and $\partial_i^1 A(\phi,\psi)$ (resp. $\partial_i^2 A(\phi,\psi)$) for $\frac{\partial}{\partial \phi_i} A(\phi,\psi)$ (resp. $\frac{\partial}{\partial \psi_i} A(\phi,\psi)$).

2.3 Solution to the lowest order RG-equations

To solve (21,22), one switches to Fourier space (in the field):

$$U^K = \int d\phi \, e^{-iK.\phi} U(\phi) \tag{27}$$

$$V^{KPx} = \int d\phi \, d\psi \, e^{-iK.\phi - iP.\psi} \, V(\phi,\psi,x) \tag{28}$$

where $K.\phi \equiv \sum_i K_i \phi_i$. It turns out that the equation for V_l can be integrated explicitely as a retarded function of U_l:

$$V_l^{KPx} = \frac{1}{2} \left(F_l^{KPx} - \delta(x) \int_y F_l^{KPy} \right) \tag{29}$$

$$F_l^{KPx} = -\int_0^l dl' \; (K.\partial G_{l'}^x.P) \, U_{l'}^K U_{l'}^P \, e^{\frac{1}{2} K.(G_l^{x=0} - G_{l'}^{x=0}).K + \frac{1}{2} P.(G_l^{x=0} - G_{l'}^{x=0}).P + K.(G_l^x - G_{l'}^x).P} \tag{30}$$

since we have chosen $V_{l=0}^{KPx} = 0$. One can then reinject this result in (21) and obtain a closed RG equation for $U_l(\phi)$:

$$\begin{aligned} \partial_l U_l(\phi) = & -\frac{1}{2} \partial G_{ij}^{x=0} \partial_i \partial_j U_l(\phi) - \frac{1}{2} \int_{KP} e^{i(K+P).\phi} \int_x K.(\partial G_l^x - \partial G_l^{x=0}).P \\ & \int_0^l dl' \; K.\partial G_{l'}^x.P \, e^{\frac{1}{2} K.(G_l^{x=0} - G_{l'}^{x=0}).K + \frac{1}{2} P.(G_l^{x=0} - G_{l'}^{x=0}).P + K.(G_l^x - G_{l'}^x).P} U_{l'}^K U_{l'}^P \end{aligned} \tag{31}$$

This is the *exact* renormalization equation for an arbitrary local interaction $U(\phi)$ to $\mathcal{O}(U^2)$. Note that the second order term is retarded in l, since as discussed above, local terms are generated only after integration.

More generally, this procedure can be carried out to any order in U using the hierarchical structure of the flow equations for p-local interactions (see Appendix C). It is found that the general structure for the flow of the local part is

$$\partial_l U_l(\phi) = \beta \left[U_{l'<l}(\phi') \right](\phi) \tag{32}$$

$$= \sum_{n \geq 1} \int_{l_1 < l_2 < \ldots < l_n < l} \mathcal{K}_{l, l_1 .. l_{n-1}}^n [\frac{\partial}{\partial \phi_1}, .. \frac{\partial}{\partial \phi_n}] U_{l_1}(\phi_1) \ldots U_{l_n}(\phi_n)|_{\phi_p = \phi} \tag{33}$$

and in (31) we achieved the calculation of the β function to second order in U. Once the solution of (32) is known up to $\mathcal{O}(U^p)$, all the p'-local flowing interactions for $p' \leq p$ are also known by injecting the solution for U_l. For example, for $p = 2$, the bilocal part is obtained from (29,30) injecting the solution to (31).

To a given order in U one can also perform a loop expansion by expanding the exponentials of propagators which appear in (31) (see Appendix C.4 for the $\mathcal{O}(U^3)$ equation). To order U^2 and one loop it reads

$$\partial_l U_l(\phi) = J_l^0 \partial_{ii} U_l(\phi) + \int_0^l dl' \, J_{ll'}^D \partial_{ij} U_{l'}(\phi) \partial_{ij} U_{l'}(\phi) \tag{34}$$

$$J_l^0 = -\frac{1}{2} \int_q \partial G_l^q \tag{35}$$

$$J_{ll'}^D = -\frac{1}{2} \int_q \partial G_l^q \partial G_{l'}^q \tag{36}$$

In order to compute the correlation functions for small wave-vectors q, the strategy of the RG consists in performing a perturbative calculation in the theory renormalized up to $l' = \ln(\Lambda_0/q)$. In the favorable cases, the interaction $\mathcal{V}_l$ flows, from a small initial initial interaction $\mathcal{V}_0$, to "fixed points" in functional space (up to appropriate rescalings) controlled by a small parameter (such as the offset from the critical dimension). Once the asymptotic large l behaviour of $U_l(\phi)$ is known, one uses the invariance property of $W(J)$ (see (11) and Appendix C.5 where this is done in details) to compute the observables. To lowest order in U, it is sufficient to keep only the local part in (13) which yields

$$W(J) = \frac{1}{2} J : G : J - \int_x \int_K U_l^K e^{-\frac{1}{2} K.G^{x=0}.K} e^{iK.(G:J)^x} + \mathcal{O}(U_l^2) \tag{37}$$

Thus one has for the two-point function $\langle \phi_i^q \phi_j^{q'} \rangle_{\mathcal{S}}^c = \delta(q + q') \mathsf{C}_{ij}^q$ with

$$\mathsf{C}_{ij}^q = G_{ij}^q + \int_K (K.G^q)_i (K.G^q)_j \hat{U}_l^K \tag{38}$$

$$\hat{U}_l^K = U_l^K e^{-\frac{1}{2} K.G_l^{x=0}.K} \tag{39}$$

and more generally the n-point function ($n \neq 2$):

$$\langle \phi_{j_1}^{q_1} \dots \phi_{j_n}^{q_n} \rangle_{\mathcal{S}}^c = -\delta_{\sum_i q_i} \left(\prod_i G_{j_i k_i}^{q_i} \partial_{k_i} \right) \hat{U}_l(0) \tag{40}$$

with $\delta_q \equiv (2\pi)^D \delta(q)$. To compute e.g. the two point correlation function at wave-vector q, one carries perturbation theory in $U_{l'}$ at a large scale l' and it is convenient to choose $l' = \ln \Lambda/q$. To first order in $U_{l'}(\phi)$, one has

$$\mathsf{C}_{ij}^q = G_{ij}^q + \int_K (K.G^q)_i (K.G^q)_j U_{l'=\ln \Lambda/q}^K e^{-\frac{1}{2} K.G_{l'=\ln \Lambda/q}^{x=0}.K} \tag{41}$$

Of course, since $W(J)$ is by construction independent of l the result should not depend on the choice of l'. Using the RG flow equation, it can be checked order by order in perturbation in powers of U_l that this is the case.

In the above computation of the two point function, the natural vertex which appear is not $U_l(\phi)$ but $\hat{U}(\phi) = \int_K e^{iK.\phi} \hat{U}_l^K$; it is thus interesting to study directly its flow equation.

3 Removing of tadpoles and application to ϕ^4

3.1 Modified Polchinski equation

It is useful for some applications, and in particular to simplify higher orders calculations, to get rid of the linear term in the Polchinski equation. This can be achieved exactly by introducing the following functional:

$$\hat{\mathcal{V}}_l(\phi) = e^{\frac{1}{2}\frac{\delta}{\delta\phi}:G_l:\frac{\delta}{\delta\phi}}\mathcal{V}_l(\phi) \tag{42}$$

Inserted in (9) one finds that it satisfies

$$\partial_l\hat{\mathcal{V}}_l(\phi) = \frac{1}{2}e^{\frac{\delta}{\delta\phi_1}:G_l:\frac{\delta}{\delta\phi_2}}\frac{\delta}{\delta\phi_1} : \partial G_l : \frac{\delta}{\delta\phi_2}\hat{\mathcal{V}}_l(\phi_1)|_{\phi_1=\phi}\hat{\mathcal{V}}_l(\phi_2)|_{\phi_2=\phi} \tag{43}$$

The graphical representation of this equation is drawn in Fig. 2. Since this equation does not contain a linear term, its solution does not contain tadpole-like diagrams. This functional has thus several advantages: first it enters directly the computation of any observable, second its flow is simpler than the one of the bare vertices. Finally in the context of quantum field theory it has the direct meaning of being the normally ordered vertices.

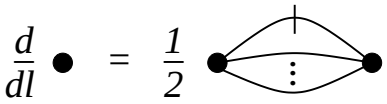

Figure 2: Graphical representation of the modified Polchinski equation. The point represent any vertex, the broken line is a propagator on shell ∂G_l and the full line is a G_l.

We now perform a multilocal expansion, similar to the one introduced in the previous Section, but on the functional $\hat{\mathcal{V}}_l(\phi)$ as:

$$\hat{\mathcal{V}}_l(\phi) = \int_x \hat{U}_l(\phi^x) + \int_{xy} \hat{V}_l(\phi^x, \phi^y, x-y) + .. \tag{44}$$

The modified Polchinski equation (43) can be solved order by order in $\hat{U}_l(\phi)$. The general analysis is performed in the Appendix C.1. Here we give only the result to order $\hat{U}^2$ which reads:

$$\partial_l\hat{U}_l(\phi) \;=\; \frac{1}{2}\int_x e^{\partial^1 G_l^x \partial^2}\partial^1\partial G_l^x\partial^2\hat{U}_l(\phi_1)|_{\phi_1=\phi}\hat{U}_l(\phi_2)|_{\phi_2=\phi} \tag{45}$$

An interesting property of this equation is that it is now local in l.

Expansion in the number of loops k, restricted to order $\hat{U}_l^2$, is thus straightforward:

$$\partial_l\hat{U}_l(\phi) = \frac{1}{2}\sum_{k=1}^{+\infty}\frac{1}{k!}I_l^k\partial_{i_1,..i_{k+1}}\hat{U}_l(\phi)\partial_{i_1,..i_{k+1}}\hat{U}_l(\phi) \tag{46}$$

$$I_l^k = \int_x (G_l^x)^k\partial G_l^x = \frac{1}{k+1}\partial_l\int_x (G_l^x)^{k+1} \tag{47}$$

Note that the multilocal expansion for $\hat{\mathcal{V}}$ and $\mathcal{V}$ are not identical, i.e they do not produce order by order equations which can be transformed back into each others. However, they should yield the same result at the end when calculating universal fixed point quantities. This property will be tested and used in the rest of the paper.

We also give the expression of the bilocal term, as one must check that it effectively reaches a fixed point for consistency. It reads as a function of the local term:

$$\hat{V}_l(\phi_1,\phi_2,q)=\frac{1}{2}\int_x(e^{iqx}-1)\int_0^l dl'\, e^{\partial^1 G^x_{l'}\partial^2}\partial^1\partial G^x_{l'}\partial^2\hat{U}_{l'}(\phi_1)\hat{U}_{l'}(\phi_2) \tag{48}$$

3.2 Modified RG equation to one loop and application to $O(n)$ model

Expanding the exponential of the propagator in (45) yields a loop expansion of the beta function to order U^2. To one loop this gives:

$$\partial_l\hat{U}_l(\phi)=I_l^D\partial_{ij}\hat{U}_l(\phi)\partial_{ij}\hat{U}_l(\phi) \tag{49}$$

$$I_l^D=\frac{1}{2}\int_q\partial G_l^q G_l^q \tag{50}$$

As a simple application let us consider the $O(n)$ model in $D=4-\epsilon$ with the polynomial interaction:

$$U_l(\phi)=\frac{1}{2}g_{2,l}\phi^2+\frac{1}{4!}g_{4,l}(\phi^2)^2+\frac{1}{6!}g_{6,l}(\phi^2)^3+.. \tag{51}$$

the dimensionless variables being $\tilde{g}_{n,l}=g_{n,l}\Lambda_l^{\frac{D-2}{2}n-D}$. The RG can be performed either using $U_l(\phi)$ as in the previous section, or in terms of $\hat{U}_l(\phi)$, by which we start. One has:

$$\hat{U}_l(\phi)=e^{\frac{1}{2}G_l^{x=0}\nabla^2}U_l(\phi)=\frac{1}{2}a_{2,l}\phi^2+\frac{1}{4!}a_{4,l}(\phi^2)^2+\frac{1}{6!}a_{6,l}(\phi^2)^3+.. \tag{52}$$

and, similarly [20] $\tilde{a}_{n,l}=a_{n,l}\Lambda_l^{\frac{D-2}{2}n-D}$ thus:

$$\tilde{a}_{2,l}=a_{2,l}\Lambda_l^{-2} \tag{53}$$

$$\tilde{a}_{4,l}=a_{4,l}\Lambda_l^{-\epsilon} \tag{54}$$

$$\tilde{a}_{6,l}=a_{6,l}\Lambda_l^{2-2\epsilon} \tag{55}$$

From $\partial_{ij}\hat{U}_l(\phi)=a_{2,l}\delta_{ij}+\frac{a_{4,l}}{3!}(\delta_{ij}\phi^2+2\phi_i\phi_j)+\frac{a_{6,l}}{5!}(\delta_{ij}(\phi^2)^2+4\phi_i\phi_j\phi^2)+..$, one obtains the RG equations:

$$\partial\tilde{a}_{2,l}=2\tilde{a}_{2,l}+I^D\frac{2}{3}(n+2)\tilde{a}_{4,l}\tilde{a}_{2,l}+\mathcal{O}(\tilde{a}_{4,l}^2) \tag{56}$$

$$\partial\tilde{a}_{4,l}=\epsilon\tilde{a}_{4,l}+I^D\frac{2}{3}(n+8)\tilde{a}_{4,l}^2+\mathcal{O}(\tilde{a}_{4,l}^3) \tag{57}$$

$$\partial\tilde{a}_{6,l}=(-2+2\epsilon)\tilde{a}_{6,l}+\mathcal{O}(\tilde{a}_{4,l}^3) \tag{58}$$

in terms of the single integral I^D which has a universal value in $D = 4$:

$$I^D = \Lambda_l^\epsilon I_l^D = \Lambda_l^\epsilon \frac{1}{2} \int_q \frac{c'(\frac{q^2}{2\Lambda_l})}{\Lambda_l^2} \frac{c(\frac{q^2}{2\Lambda_l})}{q^2} \tag{59}$$

$$= \frac{1}{2} S_D \int_{s>0} (2s)^{-\epsilon/2} c'(s) c(s) = -\frac{S_4}{4} + \mathcal{O}(\epsilon) \tag{60}$$

with $S_4 = 1/(8\pi^2)$ and here and in the following S_D is the unit sphere area divided by $(2\pi)^D$. Thus $\tilde{a}_{4,l}$ flows to the fixed value:

$$\tilde{a}_4^* = \frac{3}{(n+8)} 16\pi^2 \epsilon \tag{61}$$

which is universal to this order. We have indicated terms in the above RG equations which arise at two loops and yield fixed point for $\tilde{a}_{2,l}$ and $\tilde{a}_{6,l}$ with:

$$\tilde{a}_2^* = \mathcal{O}(\epsilon^2) \quad , \quad \tilde{a}_6^* = \mathcal{O}(\epsilon^3) \tag{62}$$

and more generally $\tilde{a}_{2n}^* = \mathcal{O}(\epsilon^n)$ for $n \geq 3$. The derivation of this is simple but goes beyond this paper [16]. This fixed point is unstable in the direction $\tilde{a}_2$ and stable in all other directions. The stability eigenvalues to $\mathcal{O}(\epsilon)$ read:

$$\lambda_2 = 2 - \frac{n+2}{n+8} \epsilon \tag{63}$$

$$\lambda_4 = -\epsilon \tag{64}$$

$$\lambda_{2n} = -2(n-2) + \epsilon(n-1) \quad n \geq 3 \tag{65}$$

The critical manifold of the $O(n)$ model corresponds to $\tilde{a}_2 = \tilde{a}_2^*$. This corresponds indeed to the massless case, since the self energy

$$\Sigma^q = \hat{U}''(0) + \mathcal{O}(\hat{U}^2) = \Lambda_l^2 \tilde{a}_{2,l} \tag{66}$$

vanishes to lowest order on the critical manifold. The instability eigenvalue at the the fixed point gives the critical exponent γ:

$$\gamma = \frac{2}{\lambda_2} = 1 + \frac{(n+2)}{2(n+8)} \epsilon + \mathcal{O}(\epsilon^2) \tag{67}$$

thus recovering the standard result. One also gets $\omega = -\epsilon$ and $\eta = \mathcal{O}(\epsilon^2)$.

Note that to first order in ϵ there is no q dependence of Σ^q and no wave function renormalization. This can be incorporated in the method to two loops (see Appendix A.2 and [16]).

4 Application to disordered elastic systems

4.1 FRG equations and universality to one loop

4.1.1 The model

Let us consider an elastic system of internal dimension D embedded in a disordered medium. It is described by a single component displacement field u^x (x is the internal variable), which is either the height function for an interface problem, or the continuous deformation field for periodic systems. The energy reads

$$H(u) = \int_x \left(\frac{c}{2} |\nabla u^x|^2 - W(x, u^x) + \frac{m^2}{2} |u^x|^2 \right) \tag{68}$$

where a short-distance cutoff is implicit. The elastic constant is set to $c = 1$ here, and the mass to $m = 0$, its effect will be studied in Section 4.3. The disordered potential $-W(x, u)$ is a random variable which has the following properties (i) $\overline{W(x, u)} = 0$ (ii) the potential at different x are uncorrelated (iii) the distribution of $W(x, u)$ is translationally invariant in u space. Its cumulants read

$$\overline{W(x_1, u_1) \dots W(x_N, u_N)}^c = \delta(x_1 - x_2) \dots \delta(x_1 - x_N) S^{(N)}(u_1, \dots, u_N) \tag{69}$$

where the symmetric functions $S^{(N)}$ satisfy $S^{(N)}(u_1, \dots, u_N) = S^{(N)}(u_1 + u, \dots, u_N + u)$ for any u. In particular, the second cumulant is denoted by $R(u - u') = S^{(2)}(u, u')$. In the case of an interface, these correlators can be either long–range, e.g. random field, or short–range, e.g. random bond. In the periodic case, the cumulants have the periodicity of the lattice. We assume parity symmetry $S^{(N)}(-u_1, \dots, -u_N) = S^{(N)}(u_1, \dots, u_N)$. This problem is usually studied by introducing n replicas ϕ^x_a, $a = 1 \dots n$, of u^x and by averaging over the disorder. It yields the action

$$\mathcal{S}(\phi) = \int_x \left[\frac{1}{2T} \sum_a |\nabla \phi^x_a|^2 - \sum_{N \geq 2} \frac{1}{N! T^N} \sum_{a_1 \dots a_N} S^{(N)}(\phi^x_{a_1}, \dots, \phi^x_{a_N}) \right] \tag{70}$$

Thus the bare $\mathcal{S}$ has the general form (3) with

$$G^q_{ab} = \frac{T}{q^2} \, c(\frac{q^2}{2\Lambda^2}) \, \delta_{ab} \tag{71}$$

$$\mathcal{V}(\phi) = \int_x U(\phi^x) \tag{72}$$

$$U(\phi) = -\frac{1}{2T^2} \sum_{ab} R(\phi_a - \phi_b) - \frac{1}{3! T^3} \sum_{abc} S^{(3)}(\phi_a, \phi_b, \phi_c) + \dots \tag{73}$$

We will consider any cutoff function $c(s)$ such that $c(0) = 1$ with no loss of generality and with $c'(0) = 0$ for convenience (see below).

Direct perturbation theory can be performed on this model. One can show that it has a well defined $T = 0$ limit (see Appendix D). Furthermore, at $T = 0$ this perturbation theory is in fact trivial, i.e the disorder average of any observable is identical to its value in the linear random force model, as shown in the Appendix D. Within the exact RG one can in fact escape this well known dimensional reduction phenomenon, since, as we will see below, the flowing disorder becomes non analytic. As shown below it yields non trivial results for correlations.

4.1.2 RG analysis

We now use the exact RG method introduced above. For now we use the RG equations based on the multilocal expansion of $\mathcal{V}$ while the other method in terms of $\hat{\mathcal{V}}$ (explained in Section 3.1 and Appendix C.1) will be used in Section 4.3. The method with $\mathcal{V}$ turns out to be more convenient to analyze finite T effects. The l dependence of the Gaussian part is implemented by the choice

$$(G_l)^q_{ab} = T(\overline{G}_l)^q_{ab} = \frac{T}{q^2}\, c(\frac{q^2}{2\Lambda_l^2})\, \delta_{ab} \tag{74}$$

where $\Lambda_l = \Lambda e^{-l}$. This choice is particularly convenient here since there is no correction to any order to the connected quadratic part (statistical tilt symmetry[21]). The flowing interaction functional $\mathcal{V}_l(\phi)$ remains translationally and parity invariant in x space. Since translation invariance in the u space is conserved, its local part $U_l(\phi)$ remains of the form (73). In order to obtain fixed points it is convenient to define a rescaled dimensionless temperature $\tilde{T}_l = T\Lambda_l^{D-2}$ and rescaled functions

$$\begin{aligned}
\tilde{U}_l(\phi) &= \Lambda_l^{-D} U_l(\phi) && (75)\\
&= -\frac{1}{2\tilde{T}_l^2}\sum_{ab}\tilde{R}_l(\phi_a-\phi_b) - \frac{1}{3!\tilde{T}_l^3}\sum_{abc}\tilde{S}^{(3)}(\phi_a,\phi_b,\phi_c)+\dots && (76)\\
\tilde{R}_l(u) &= \Lambda_l^{D-4} R_l(u) && (77)\\
\tilde{S}_l^{(N)}(u_1,\dots,u_N) &= \Lambda_l^{DN-2N-D} S_l^{(N)}(u_1,\dots,u_N) && (78)
\end{aligned}$$

The general RG equation (32) for $U_l(\phi)$ implies a set of flow equations for the rescaled cumulants $\tilde{R}_l(u)$, $\tilde{S}_l^{(N)}(u_1,\dots,u_N)$ (since the former is in fact a *set* of equations for functions of a n-dimensional vector ϕ for any n). The rescalings above have been chosen such that these rescaled functions flow to fixed functions denoted $\tilde{R}^*(u)$, $\tilde{S}^{(N)*}(u_1,\dots,u_N)$ independent of T.

An important property of the theory (70) is that it admits a well-defined $T=0$ limit, at least at the perturbative level. This can be seen either by examination of the diagrammatics (all negative powers of T in the perturbative calculation of observables are in factor of a positive power of n, see Appendix D), or in the $T=0$ dynamics [15]. Similarly there exists a well-defined $T=0$ limit of the set of flow equations for the cumulants. For small $\epsilon = 4-D$, this complicated set of coupled equations can be organized in powers of ϵ. Specifically one finds that $\tilde{R}^* = \mathcal{O}(\epsilon)$ while $\tilde{S}^{(N)*} = \mathcal{O}(\epsilon^N)$ for $N \geq 3$. This can be seen on the schematic structure (ordered in U and T) of the flow equations obeyed by the rescaled $\tilde{R}_l$, $\tilde{S}_l^{(N)}$, which can be read off from Appendix C.4:

$$\partial U = TU + T^2U^2e^T + T^3U^3e^T + \dots \tag{79}$$

The two first terms reproduce (31) since $\partial G \propto T$, while the third term mimics the $\mathcal{O}(U^3)$ in the β function. Its three U vertices must be linked by at least three propagators because of the constraint of locality. Substituting symbolically $\tilde{U} = \tilde{R}/T^2 + \tilde{S}/T^3$, where we restrict to the two lowest cumulants, one finds

$$\begin{aligned}
\partial\tilde{R} &= \epsilon\tilde{R} + \tilde{S}|_2 + \tilde{R}^2|_2 + \tilde{R}\tilde{S}e^T/T|_2 + \tilde{R}^3e^T/T|_2 && (80)\\
\partial\tilde{S} &= (6-2D)\tilde{S} + \tilde{R}\tilde{S}|_3 + \tilde{R}^3|_3 + \tilde{S}^2e^T/T|_3 + \tilde{R}^2\tilde{S}e^T/T|_3 && (81)
\end{aligned}$$

where we have denoted projections on 2 and 3 replica parts. All terms containing $1/T$ vanish after these projections since a well-defined $T = 0$ limit exists. We have discarded terms, such as $\partial_l \tilde{S} = T\tilde{R}^2|_3$, which (formally) vanish at $T = 0$.

One sees immediately on these equations that the fixed $\tilde{R}^* = \mathcal{O}(\epsilon)$ while $\tilde{S}^* = \mathcal{O}(\epsilon^3)$. This can be generalized by noting that the lowest order (in ϵ) correction to $S^{(N)}$ is of the form $\tilde{R}^N|_N$ thus $\tilde{S}^{(N)*} = \mathcal{O}(\epsilon^N)$. To $\mathcal{O}(\epsilon^2)$ at $T = 0$ we thus need:

$$\partial\tilde{R} = \epsilon\tilde{R} + \tilde{R}^2|_2 + \tilde{S}|_2 + \tilde{R}^3 e^T/T|_2 \tag{82}$$

$$\partial\tilde{S} = (6 - 2D)\tilde{S} + \tilde{R}^3|_3 \tag{83}$$

In this paper we simply perform the $\mathcal{O}(\epsilon)$ calculation to which we now turn, for which consideration of two replica terms is sufficient.

We perform the analysis in the $T = 0$ limit as explained above. The propagator can be expressed in terms of dimensionless quantities as $G_l^x = \tilde{T}_l \int_q \frac{c(\frac{q^2}{2})}{q^2} e^{iq\Lambda_l x}$. At finite T, the exponentials of propagators in (31) would reduce to 1 asymptotically at large l. This is also true in the $T = 0$ limit for any l. It is thus a priori unnecessary to include higher number of loops within order U^2.

Denoting by $\tilde{M}_l(\phi)$ the two–replica term contained in the local operator U

$$\tilde{M}_l(\phi) = -\frac{1}{2}\sum_{ab} \tilde{R}_l(\phi_a - \phi_b) \tag{84}$$

the flow equation to one loop (34) (using the change of variables $l' \to l - l'$ yields

$$\partial_l \tilde{M}_l(\phi) = (4 - D)\tilde{M}_l(\phi) - \frac{1}{2}\int_0^l dl'\, \mathsf{K}_{l'} \sum_{ab} [\partial_{ab}\tilde{M}_{l-l'}(\phi)]^2|_{2-rep} \tag{85}$$

where the kernel responsible for the retarded nature of the flow is

$$\mathsf{K}_{l'} = 4J^D_{l,l-l'}\Lambda^{2\epsilon}_{l-l'}\Lambda_l^{-\epsilon} = 2e^{-(6-D)l'}\int_q c'(\frac{q^2}{2})c'(\frac{q^2}{2}e^{-2l'}) \tag{86}$$

Since $c'(u)$ is typically peaked around $u \sim 1$ and decreases fast at infinity, one sees on (86) that the range of the kernel K_l is also of order one and can be made as small as desired by choosing narrow enough cutoff functions. The above RG equation (85) involves computing the contraction:

$$\sum_{ab}\left[\partial_a\partial_b\tilde{M}(\phi)\right]^2 = \sum_{ab}\left[\tilde{R}''(\phi_a - \phi_b)^2 - 2\tilde{R}''(0)\tilde{R}''(\phi_a - \phi_b)\right] \tag{87}$$

$$+\sum_{abc}\tilde{R}''(\phi_a - \phi_c)\tilde{R}''(\phi_c - \phi_b) \tag{88}$$

where we have used $\partial_a\partial_b\tilde{M}(\phi) = \delta_{ab}\sum_c \tilde{R}''(\phi_a - \phi_c) - \tilde{R}''(\phi_a - \phi_b)$. The last sum being a three replica term, it does not enter the equation for $\tilde{R}$ (it is a correction to S proportional to T), which reads:

$$\partial_l\tilde{R}_l(u) = \epsilon\tilde{R}_l(u) + \int_0^l dl'\, \mathsf{K}_{l'}\left(\frac{1}{2}\tilde{R}''_{l-l'}(u)^2 - \tilde{R}''_{l-l'}(0)\tilde{R}''_{l-l'}(u)\right) \tag{89}$$

Let us first study the case of periodic elastic systems, with $\tilde{R}_l(\phi)$ periodic of period 1. Taking the large l limit we find the fixed point equation:

$$0 = \epsilon \tilde{R}^*(u) + (\int_0^{+\infty} dl' \, \mathsf{K}_{l'}) \left(\frac{1}{2} \tilde{R}^{*\prime\prime}(u)^2 - \tilde{R}^{*\prime\prime}(0) \tilde{R}^{*\prime\prime}(u) \right) \tag{90}$$

It is now easy to see that the factor $\mathsf{K} = \int_0^\infty dl \, \mathsf{K}_l$ in (90), which a priori depends on the dimension of space and of the whole arbitrary cutoff function $c(s)$, becomes universal in $D = 4$. Indeed:

$$\begin{aligned} \mathsf{K} &= 2 S_D 2^{-\epsilon/2} \int_0^\infty ds \, s^{-\epsilon} c'(s) \int_0^s dt \, t^\epsilon c'(t) & (91) \\ &= 2 S_4 \int_0^\infty ds \, c'(s)(c(s) - 1) + \mathcal{O}(\epsilon) = S_4 + \mathcal{O}(\epsilon) & (92) \end{aligned}$$

where we used the new variables $s = q^2/2$, then $t = s e^{-2l}$, kept only the lowest order in ϵ, and used $c(0) = 1$. We denote by S_D the surface of the unit sphere in D dimensions divided by $(2\pi)^D$. Thus, to one loop, the FRG equation *does not depend on the cutoff procedure*. It coincides with the fixed point equation obtained[14, 22] from Wilson's momentum shell renormalization.

In Appendix C.3, we also mention the result of a two-loop calculation of the beta-function in our exact renormalization framework.

The solution to (90) is known to be the 1–periodic function defined by

$$\tilde{R}^*(u) = \frac{\epsilon}{72 S_4} \left(\frac{1}{36} - u^2 (1-u)^2 \right) \tag{93}$$

for $0 < u < 1$. This fixed point function is non–analytic which is an important and unusual feature. It was argued in [14] that this non–analyticity appears at a finite scale. This scale $R_c = e^{l_c}/\Lambda$ can be identified with the Larkin length at which metastability and glassiness appears. Taking the fourth derivative at $u = 0$ of (89) yields a closed retarded equation for $\tilde{R}_l^{iv}(0)$

$$\partial_l \tilde{R}_l^{iv}(0) = \epsilon \tilde{R}_l^{iv}(0) + 3 \int_0^l dl' \, \mathsf{K}_{l'} \tilde{R}_{l-l'}^{iv}(0)^2 \tag{94}$$

In the limit of narrow cutoffs, the equation becomes local and $\tilde{R}_l^{iv}(0)$ diverges at a finite scale. One can show that this feature persits in the non–local equation.

The case of an interface (i.e a directed polymer for $D = 1$) corresponds to another fixed point where one must rescale the function $\tilde{R}_l(u)$ as follows:

$$\tilde{R}_l(u) = e^{4\zeta l} r_l(u e^{-\zeta l}) \tag{95}$$

and we must now determine $\zeta = \mathcal{O}(\epsilon)$ such that $r_l(v)$ converges to a fixed point $r^*(v)$. Inserting (95) into (89) yields:

$$\partial_l r_l(v) = (\epsilon - 4\zeta) r_l(v) + \zeta v r_l'(v) + \int_0^l dl' \, \mathsf{K}_{l'} e^{-4\zeta l'} \left(\frac{1}{2} r''_{l-l'}(v e^{\zeta l'})^2 - r''_{l-l'}(0) r''_{l-l'}(v e^{\zeta l'}) \right) \tag{96}$$

Although the kernel has been modified, this does not affect the results for the fixed point to lowest order in ϵ. The fixed point equation reads:

$$0 = (\epsilon - 4\zeta) r^*(v) + \zeta v r^{*\prime}(v) + S_4 \left(\frac{1}{2} r^{*\prime\prime}(v)^2 - r^{*\prime\prime}(0) r^{*\prime\prime}(v) \right) \tag{97}$$

and is thus universal, independent of $c(s)$. This shows that ζ, which, as shown below is the roughness exponent, is universal to one loop. It will be studied below for the random field case and in the case of short range disorder it is thus equal to $\mathcal{O}(\epsilon)$ to the values given in [14].

4.1.3 Correlation function

Let us now compute the two–point correlation function at $T = 0$ using (12). To lowest order in ϵ, it is sufficient to use the first order formula (41). The bare Gaussian part G^q vanishes at $T = 0$. We thus get:

$$\mathsf{C}^q_{ab} = -\frac{T^2}{q^4} c(\frac{q^2}{2\Lambda^2})^2 \partial_a \partial_b U_l(\phi)|_{\phi \equiv 0} = -\frac{R''_{l=\ln(q/\Lambda)}(0)}{q^4} = -\frac{\tilde{R}^{*\prime\prime}(0)}{q^D} = \frac{\epsilon}{S_4} \frac{1}{36} q^{-D} \tag{98}$$

where we used that $\tilde{R}_l$ converges to the fixed point $\tilde{R}^*$ and small q such that $c(q^2/2\Lambda^2) = 1$. In real space, it yields logarithmic growth of the displacements with a universal prefactor

$$\overline{\langle (u^x - u^0)^2 \rangle} \;\; = \;\; \frac{\epsilon}{18} \ln |\Lambda x| \tag{99}$$

In the case of short range disorder (e.g. random bond an for Ising interface) one gets instead:

$$\mathsf{C}^q_{ab} = -\frac{R''_{l=\ln(q/\Lambda)}(0)}{q^4} = -e^{2\zeta l} \Lambda_l^\epsilon \frac{r^{*\prime\prime}(0)}{q^D} \sim \frac{1}{q^{D+2\zeta}} \tag{100}$$

This yields to a roughness exponent $\overline{\langle (u^x - u^0)^2 \rangle} \sim |x|^{2\zeta}$ with a nonuniversal amplitude (since the FRG fixed point equation (97) is invariant under $r^*(v) \to \lambda^4 r^*(v/\lambda)$ and, contrarily to the periodic case, nothing here fixed the scale).

We can now investigate in more details the structure of the asymptotic flow of the various higher order interactions (three replica terms and higher, as well as bilocal interaction and more). Although this is beyond the scope of this paper, such an analysis is in principle necessary for consistency, i.e to ensure the existence of a global fixed point (for all interactions) and the validity of the result to $\mathcal{O}(\epsilon)$. We sketch it here for the periodic case $\zeta = 0$, generalizations to interfaces being simple.

We start with estimating the higher cumulants of the renormalized disorder, i.e the higher replica components of the local interaction U_l. To lowest order in R, the correction to $S_l^{(N)}$ is proportional to R_l^N and takes the schematic form:

$$\partial \tilde{S}^{(N)} = -(2N + D - DN) \tilde{S}^{(N)} + \tilde{R}''^N \tag{101}$$

Graphically, the diagram is made of one loop. We dropped the numerous higher order terms in ϵ coming from contractions of various other cumulants than R. One finds that the fixed

point $\tilde{S}^{(N)*}$ takes the following form to lowest order in ϵ

$$\sum_{a_1 \dots a_N} \tilde{S}^{(N)*}(\phi_{a_1} \dots \phi_{a_N}) = c_{N,D} \left(\mathrm{Tr}\left(W^N\right) - \sum_{a_1 \dots a_N, b} \tilde{R}^{*\prime\prime}_{b a_1} \dots \tilde{R}^{*\prime\prime}_{b a_N} \right) \tag{102}$$

where $W_{ab}(\phi) = \delta_{ab} \sum_c \tilde{R}^{*\prime\prime}_{ac} - \tilde{R}^{*\prime\prime}_{ab}$, $\tilde{R}^{*\prime\prime}_{ab}$ denotes $\tilde{R}^{*\prime\prime}(\phi_a - \phi_b)$, and $c_{N,D}$ is some number depending on the cutoff procedure. The last term in the trace has been substracted since the product of the N δ's is a $N+1$ replica term. For instance, the third cumulant is of order $\mathcal{O}(\epsilon^3)$ and reads

$$\tilde{S}^{(3)*}(u_1, u_2, u_3) = c_{3,D} \mathrm{Sym}_{u_1,u_2,u_3} [\tilde{R}^{*\prime\prime}(u_2 - u_3) \tilde{R}^{*\prime\prime}(u_3 - u_1) \tilde{R}^{*\prime\prime}(u_1 - u_2) \tag{103}$$

$$-3\tilde{R}^{*\prime\prime}(u_1 - u_2) \tilde{R}^{*\prime\prime}(u_1 - u_3)^2 + 3\tilde{R}^{*\prime\prime}(0) \tilde{R}^{*\prime\prime}(u_2 - u_1) \tilde{R}^{*\prime\prime}(u_3 - u_1)] \tag{104}$$

where $c_{3,4}$ is computed in [16] and reads:

$$c_{3,4} = \frac{S_4}{12} \int_0^\infty ds \, \frac{(1 - c(s))^3}{s^2} \tag{105}$$

Now we check that the bilocal part has a well defined fixed point. Its expression is given by (29), where at $T = 0$, the exponentials should be expanded at most to first order. The zero-th order term yields three replica terms, while the first order term yields two replica terms (as well as a correction proportional to T to three replica terms which we can discard at $T = 0$). Thus we get

$$V_l(\phi_1, \phi_2, q) = -\frac{\Lambda_l^\epsilon}{2T^2} \sum_{ab} \tilde{V}_l^{(2,2)}(\frac{q}{\Lambda_l}) - \frac{\Lambda_l^{2\epsilon-2}}{6T^3} \sum_{abc} \tilde{V}_l^{(2,3)}(\frac{q}{\Lambda_l}) \tag{106}$$

where we have explicitly separated two and three replica terms respectively:

$$\begin{aligned}
\tilde{V}_l^{(2,2)}(\tilde{q}) &= \frac{1}{2} \int_k (\frac{1}{k^2 (k+\tilde{q})^2} (c(k^2/2) - 1)(c((k+\tilde{q})^2/2) - 1) - \text{idem q=0}) \\
&\quad (\sum_{ab} \tilde{R}^{\prime\prime 1}_{ab} \tilde{R}^{\prime\prime 2}_{ab} - \tilde{R}^{\prime\prime}(0) \sum_{ab} (\tilde{R}^{\prime\prime 1}_{ab} + \tilde{R}^{\prime\prime 2}_{ab})) \\
&\quad -\frac{1}{4} (\int_k \frac{1}{k^2} c(k^2/2)) \int_0^1 d\alpha (\frac{1}{\alpha} - 1) c'(\alpha \tilde{q}^2/2) (2 \sum_{abc, a \neq b a \neq c} \tilde{R}^{\prime\prime\prime 1}_{ab} \tilde{R}^{\prime 2}_{ac} + \tilde{R}^{\prime 1}_{ab} \tilde{R}^{\prime\prime\prime 2}_{ac}) |_{2rep} \\
\tilde{V}_l^{(2,3)}(\tilde{q}) &= -3 \frac{c(\tilde{q}^2/2) - 1}{\tilde{q}^2} \sum_{abc} \tilde{R}^{\prime 1}_{ab} \tilde{R}^{\prime 2}_{ac}
\end{aligned}$$

where $\tilde{R}^{\prime 1}_{ab}$ stands for $\tilde{R}'(\phi_{1,a} - \phi_{1,b})$ etc.. The bilocal term thus has a scale invariant fixed form of order ϵ^2 and is a well-defined function of $\tilde{q} = q/\Lambda_l$ with no divergences.

More generally, we conjecture that there is a fixed asymptotic form for all multilocal interactions $V^{(p)}$ which can be explicitly written as a sum of properly rescaled multi-replica terms as

$$V_l^{(p)}(\phi_1, ..\phi_p, x_1, ..x_p) = \Lambda_l^{Dp} \sum_{c \geq 2} \frac{\Lambda_l^{c(2-D)}}{c! T^c} \sum_{a_1, ..a_c} \tilde{V}_l^{(p,c)}(\{\phi_{\alpha, a_i}\}_{i=1,..c}^{\alpha=1,..p}, \Lambda_l x_1, ..\Lambda_l x_p) \tag{107}$$

where the number of replicas c corresponds graphically to the number of connected components. The consistency of the method demands that the $\tilde{V}_l$ flow to well defined fixed points, perturbative in ϵ. It is indeed natural to conjecture that in this theory there is no wave function renormalization.

We can now come back to the calculation of the correlation function. Although for convenience we have computed C^q_{ab} from the theory with $l = \ln(q/\Lambda)$, this is unnecessary. As discussed above, the existence of a fixed point with well-defined functions of $\tilde{q} = q/\Lambda_l$ implies that the ϵ expansion of C^q_{ab} at fixed $q \ll \Lambda$ should be of the form:

$$\mathsf{C}^q_{ab} = \frac{1}{q^4}\Lambda_l^\epsilon \tilde{\Sigma}(\tilde{q}) \tag{108}$$

where the dimensionless self energy $\tilde{\Sigma}(\tilde{q})$ depends only on $\tilde{q}$ and ϵ. Since C^q_{ab} is l independent, it implies, taking the derivative, that $\tilde{\Sigma}(\tilde{q}) = C_\epsilon \tilde{q}^\epsilon$ and thus

$$\mathsf{C}^q_{ab} = C_\epsilon q^{-D} \tag{109}$$

$$C_\epsilon = \frac{2\pi^2}{9}\epsilon \tag{110}$$

to lowest order in ϵ, as in [23]. The form (109) should be valid to all orders in ϵ if the hypothesis about the fixed point formulated above are satisfied.

4.2 FRG at finite temperature

4.2.1 Renormalization equations

Since in (31) the terms in the exponentials containing the temperature go to zero as Λ_l^{D-2} one can first study the effect of temperature, compared to $T = 0$ by looking at the linear term. Up to order TR and R^2 (i.e to one loop) the RG equation thus reads:

$$\partial_l \tilde{R}_l(u) = \epsilon \tilde{R}_l(u) + \hat{T}_l \tilde{R}_l''(u) + \int_0^l dl' \, \mathsf{K}_{l'} \left(\frac{1}{2}\tilde{R}''_{l-l'}(u)^2 - \tilde{R}''_{l-l'}(0)\tilde{R}''_{l-l'}(u) \right) \tag{111}$$

with

$$\hat{T}_l = -\partial G_l^{x=0} = -T \int_q \Lambda_l^{-2} c'(\frac{q^2}{2\Lambda_l^2}) \tag{112}$$

$$= 2TS_4\Lambda_l^2 \int_{s>0} c(s) + \mathcal{O}(\epsilon) \tag{113}$$

In the case of sharp cutoff this equation has been studied in [25, 26]. It was found that at fixed u, $\tilde{R}_l(u)$ converges to $\tilde{R}^*(u)$ but that temperature rounds the cusp of the $T = 0$ solution in a boundary layer of size $u \sim \hat{T}_l$. As in [26] we look for a solution of the form:

$$\tilde{R}_l(0) - \tilde{R}_l(u) = -\tilde{R}_l''(0)\frac{u^2}{2} - \mathsf{K}\frac{\hat{T}_l^3}{\epsilon^2\chi^2} H_l(\frac{u\epsilon\chi}{\hat{T}_l}) \tag{114}$$

Here, χ (of order ϵ^0) is defined by $\epsilon\chi = \tilde{R}^{*\prime\prime\prime\prime}(0^+)$ and thus $H^{*\prime\prime\prime\prime}(0) = -1$. One has also $H_l(0) = H_l''(0) = 0$. Injecting (114) into (111) and identifying the leading order in $\hat{T}_l$, one gets:

$$\frac{x^2}{2} = H^{*\prime\prime}(x) + \frac{1}{2}\int_{l>0} K_l e^{4l} H^{*\prime\prime}(xe^{-2l})^2 \tag{115}$$

with $\int_{l>0} K_l = 1$. This equation can be solved iteratively in n for $H^*(x) = \sum_{n>0} a_n x^{2n}/(2n)!$. One has $a_1 = 1$, $a_2 = -3$ but higher a_n's are non-universal. The large x behavior of H^* is universal and given by $H^*(x) \sim x$. In the case of sharp cutoff one recovers [26]

$$H^{*\prime\prime}(x) = \sqrt{1+x^2} - 1 \tag{116}$$

This result should be further examined by consideration of consistency within higher loop corrections, which goes beyond this paper.

The most important result of the section is that the following relation between the finite temperature solution and the $T = 0$ solution:

$$\lim_{l\to+\infty} \hat{T}_l \tilde{R}_l''''(0) = C\tilde{R}_l^{*\prime\prime\prime}(0^+)^2 \tag{117}$$

where $C = S_4$ holds irrespective of the cutoff function, and thus is determined by the $T = 0$ fixed point. This property will be used below.

4.2.2 Calculation of universal susceptiblity fluctuations

It was noted recently that a signature of glassy behaviour in a disordered system was the large sample to sample fluctuations of the response to external perturbations [27, 29, 30]. These are described by the following suceptibility:

$$\chi = \frac{1}{T}\frac{1}{L^D}\int_{xy} (\langle \partial_\alpha u^x \partial_\alpha u^y\rangle - \langle \partial_\alpha u^x\rangle\langle \partial_\alpha u^y\rangle) \tag{118}$$

in a finite system of size L, which measures the response in a given sample to a field coupling to ∇u (e.g. the tilt or compression a flux lattice, or the compression response). The $\langle X\rangle$ denotes the thermal averages in a given sample. These have been studied in connection with mesoscopic behaviour of disordered systems [31]. Here we have considered only the trace of the response tensor (extension being straightforward). To perform the calculation in the replicated theory we define:

$$C_{ab}^{\alpha\beta} = \frac{1}{T}\frac{1}{L^D}\int_{xy} \langle \partial_\alpha u_a^x \partial_\beta u_b^y\rangle \tag{119}$$

$$C_{abcd} = \frac{1}{T^2}\frac{1}{L^{2D}}\int_{xyzt} \langle \nabla u_a^x \cdot \nabla u_b^y \nabla u_c^z \cdot \nabla u_d^t\rangle \tag{120}$$

We now compute respectively the first and second moment of the sample to sample fluctuations of the susceptibility. They read:

$$\overline{\chi} = C_{aa} - C_{ab} = 1 \tag{121}$$

$$\overline{\chi^2} = C_{aabb} + C_{abcd} - 2C_{aabc} \tag{122}$$

where $\overline{X}$ denote disorder averages and a, b, c, d take values all distinct from each other. Note the well known property [21] that the average susceptibility is identical to the susceptibility of the pure system. We now compute C_{abcd} to lowest order in ϵ. Only zero and one loop graphs involving respectively $R_l''(0)$ and $R_l''''(0)$ contribute. Interestingly, due to the quadratic nature of the term proportional to $R_l''(0)$ the zero loop graphs cancel in $\overline{\chi^2}$, as can be easily seen since $C_{abcd} = C_{ab}^2 + 2\sum_{\alpha\beta}(C_{ab}^{\alpha\beta})^2$ for any gaussian theory (performing the Wick contractions). One is left with:

$$C_{abcd} = (n\delta_{abcd} - (\delta_{bcd} + \delta_{cda} + \delta_{dab} + \delta_{abc}) + (\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}))AR_l''''(0) \quad (123)$$

where $A = L^{-2D}\int_w (\int_{xy} \nabla G^{x-w} \cdot \nabla G^{y-w})^2$ and thus

$$\overline{(\Delta\chi)^2} \equiv \overline{\chi^2} - \overline{\chi}^2 = AR_l''''(0) \sim R_l''''(0) C L^{4-D} \quad (124)$$

for a system of finite size L (see also [30] for a similar result in straight perturbation theory). Note that one can equivalently study the perturbation of an infinite system (i.e $L \to \infty$ first) by a periodic external field of wavevector q_{ext}. In that case $A = q_{\rm ext}^{D-4}$. Thanks to the exact RG equations at finite T and substituting $l = \ln L$ we obtain the mesoscopic susceptibility fluctuations at low temperature as:

$$\overline{(\Delta\chi)^2} = C' \frac{L^\theta}{T} \quad (125)$$

where $\theta = D - 2 + 2\zeta$ is the energy fuctuation exponent and $C' = \mathcal{O}(\epsilon^2)$ for a periodic system ($\zeta = 0$) and $C' = \mathcal{O}(\epsilon^{4/3})\sigma^{2/3}$ for an interface in random field disorder (see Section 4.3).

This result, derived here through exact FRG calculation, is consistent with the droplet picture [32]. Indeed the second moment of the susceptibility fluctuations is dominated by the rare configurations of disorder (of probability $p_{\rm deg} \ll 1$) with two almost degenerate (i.e within $\mathcal{O}(T)$ in energy) ground states as follows:

$$\overline{(\Delta\chi)^2} \sim p_{\rm deg} (\delta\chi_{\rm typ})^2 \quad (126)$$

where $p_{\rm deg} \sim T/L^\theta$ and the typical fluctuation is $\delta\chi_{\rm typ} \sim T^{-1} L^{-D} L^{2D-2+2\zeta}$ from 118. One thus recovers the above result since $\theta = D - 2 + 2\zeta$.

4.3 Interface in a biased random field and toy model

In this Section we study the model (68) in the presence of a mass term $m > 0$, which confines the fluctuations of the displacement u^x. We consider two cases (i) random field disorder (ii) periodic disorder. A physical realization of (i) consists in a domain wall separating the $\pm$ phases in a ferromagnet, submitted to a random magnetic field. The magnetic energy of the interface, assumed without overhangs, is:

$$\mathcal{E}(u) = 2\int d^D x \int_0^{u^x} du' h(u', x) \quad (127)$$

Thus the effect of the mass term corresponds to applying an additional field gradient $h(u, x) \to h(u, x) + m^2 u/2$. Note that this field gradient can either stabilize ($m^2 > 0$)

or destabilize ($m^2 < 0$) the domain wall. We will study the approach to the critical value $m^2 \to 0^+$. The case (ii) is of interest when studying the competition between disorder and e.g. a periodic potential. In the phase where the periodic potential is relevant it is natural to approximate it by replacing it by an harmonic well (see e.g. [33]).

Here we examine only ground state properties (zero temperature). We show that the disorder induced fluctuations of the displacement u^x is described, as $m \to 0$ by a universal scaling function of the form:

$$\overline{u^q u^{-q}} = m^{-\alpha} F[cq^2/m^2] \tag{128}$$

which we determine to lowest order in $\epsilon = 4 - D$. Note that c can be measured from the thermal connected correlation, which is unchanged by disorder for models like (68) which possess the statistical tilt symmetry.

4.3.1 RG equations in presence of a mass

In this Section it is more convenient to use the RG equation resulting from the multilocal expansion on $\hat{\mathcal{V}}$, which is local in l to this order. Since we are studying $T = 0$ we set $\hat{R}_l = R_l$ in the following. The RG equation reads:

$$\partial_l R_l(u) = J_l \left(\frac{1}{2} R_l''(u)^2 - R_l''(0) R_l''(u) \right) \tag{129}$$

$$J_l = -2 \int_q \partial \overline{G_l^q} \overline{G_l^q} \tag{130}$$

Using the action $\mathcal{S}_l$, the $T = 0$ correlation function reads to lowest order in R:

$$\overline{u^q u^{-q}} = -R_l''(0) \left(\frac{c(q^2/2\Lambda^2)}{cq^2 + m^2} \right)^2 \tag{131}$$

It is easy to transform (129) into the RG equation in the absence of a mass. Using the change of variable:

$$R_l(u) = c^2 \Lambda_{t(l)}^{\epsilon - 4\zeta} \tilde{R}_{t(l)}(u \Lambda_{t(l)}^{\zeta}) \tag{132}$$

one finds that $\tilde{R}_l$ satisfies the $m = 0$ flow equation:

$$\partial_t \tilde{R}_t(x) = (\epsilon - 4\zeta) \tilde{R}_t(x) + \zeta x \tilde{R}_t'(x) + S_D \left(\frac{1}{2} \tilde{R}_t''(x)^2 - \tilde{R}_t''(0) \tilde{R}_t''(x) \right) \tag{133}$$

provided that $t(l)$ satisfies

$$\Lambda_{t(l)}^{-\epsilon} \frac{dt}{dl} = J_l \frac{c^2}{S_D} \tag{134}$$

which is integrated into:

$$\frac{e^{\epsilon t_l} - 1}{\epsilon} = \int_0^\infty q^{D-1} dq \, \frac{c(q^2/2)^2 - c(q^2 e^{2l}/2)^2}{(q^2 + \alpha)^2} \tag{135}$$

with $\alpha = \frac{m^2}{c\Lambda^2}$. The function $t(l)$ is increasing and bounded. Its limit $t(+\infty) = t_\infty$ for $D < 4$ is given by:

$$\frac{e^{\epsilon t_\infty} - 1}{\epsilon} = \int_0^\infty q^{D-1} dq \, \frac{c(q^2/2)^2}{(q^2+\alpha)^2} \tag{136}$$

and it diverges for $m \to 0$ as:

$$e^{\epsilon t_\infty} \sim \epsilon \int_0^\infty q^{D-1} dq \, \frac{1}{(q^2+\alpha)^2} \sim \alpha^{-\frac{\epsilon}{2}} (1 - \frac{\epsilon}{2}) \frac{\frac{\epsilon\pi}{2}}{\sin\frac{\epsilon\pi}{2}} \tag{137}$$

In $D = 4$ it diverges as:

$$t_\infty \sim \frac{1}{2} \ln(\frac{1}{\alpha}) \tag{138}$$

We now distinguish the two cases.

4.3.2 Random field

In that case the correlations of the potential are $\overline{(W(r,u) - W(r',u'))^2} = -2\delta^D(r-r')R(u-u')$ with (from (127), $R(u) \sim -\sigma|u|$ at large u. In the massless case it is known that the FRG to one loop reproduces the purely dimensional result $u_0 u_r \sim \sigma^{2/3} c^{-4/3} r^{2(4-D)/3}$ with a roughness exponent $\zeta = (4-D)/3$. From this we expect, in the massive case, the small m behaviour:

$$u \sim \sigma^{1/3} m^{-\epsilon/3} c^{-D/6}$$

It is known [14] that one must fix $\zeta = \epsilon/3$ to obtain a reasonable fixed point. From the above equation, the (reduced) correlator of the force $\Delta_t(x) = -\tilde{R}_t''(x) S_D$ then satisfies

$$\partial_t \Delta_t(x) = \frac{\epsilon}{3} \left(x \Delta_t(x)\right)' - \frac{1}{2} \left(\Delta_t(x) - \Delta_t(0)\right)^{2''}$$

and flows for $t \to \infty$, to a fixed point $\Delta_\infty(x)$ given in terms of a function $y(x)$ [14] implicitly defined as [26]:

$$\left\{ \begin{array}{rcl} \Delta_\infty(x) & = & \Delta_\infty(0) y(x \sqrt{\frac{\epsilon}{3\Delta_\infty(0)}}) \\ \Delta_\infty(0) & = & \left(\frac{\epsilon}{24\gamma^2}\right)^{1/3} \left(\int_{-\infty}^{+\infty} dx \, \Delta_0(x)\right)^{2/3} \\ \frac{x^2}{2} & = & y(x) - 1 - \ln y(x) \end{array} \right.$$

with $\gamma = \int_0^1 dy \, \sqrt{y - 1 - \ln y} = 0.5482228893..$ Note that $\int_{-\infty}^{+\infty} dx \, \Delta_t(x)$ is t independent and thus equal to $\int_{-\infty}^{+\infty} dx \, \Delta_0(x) = -\frac{S_D}{c^2} \int R_0'' = 2\sigma \frac{S_D}{c^2}$.

Putting this together with (131), (137) yields the result:

$$\begin{aligned} &\overline{u^q u^{-q}} = \sigma^{2/3} m^{-D-2\zeta} c^{D/6} F_D[cq^2/m^2] \\ &F_D[x] = C_D \frac{1}{(1+x)^2} + \text{h.o.t} \\ &C_D = \left(\frac{(4\pi)^{D/2}}{6\Gamma(\frac{\epsilon}{2})\gamma^2} \right)^{1/3} \end{aligned} \tag{139}$$

with $\zeta = \epsilon/3$. Note that the universal scaling function must behave at large x as $F_D[x] \sim x^{\zeta - D/2}$, and is determined here only to order 0 in ϵ.

From this one also finds the local fluctuation:

$$\overline{(u^x)^2} = \epsilon^{-2/3} 6^{-1/3} \left(\frac{\epsilon \Gamma(\frac{\epsilon}{2})}{(4\pi)^{D/2}\gamma} \right)^{2/3} \sigma^{2/3} m^{-2\zeta} c^{-D/3} \tag{140}$$

which is also universal. The fact that this quantity is dominated by large scale fluctuations can be seen from the convergence of the integral ($\zeta > 0$).

The calculation can also be performed exactly in $D = 4$. One finds:

$$\overline{u^q u^{-q}} = \sigma^{2/3} c^{2/3} m^{-4} \left(\ln \frac{1}{m} \right)^{-1/3} \left(\frac{4\pi^2}{3\gamma^2} \right)^{1/3} \frac{1}{(1 + \frac{cq^2}{m^2})^2} \tag{141}$$

The values we find for C_D, which vanishes as $C_D \sim 3.5246\, \epsilon^{2/3}$ as $D \to 4^-$ are as follows $C_3 \approx 2.40653$, $C_2 \approx 1.91006$, $C_1 \approx 1.30416$, $C_0 \approx 0.82157$, remarkably close to the exact result in $D = 0$ [24] $C_0 = 1.05423856519$.

It is also useful to compare these results with the Gaussian variational method with replica symmetry breaking. Extending the calculation of [27] to the non zero mass case, which is done in Appendix E, we find the same form as (139) with

$$F_D[x] = C'_D [\frac{1}{(1+x)^2} + \frac{\theta(2-\theta)}{4} \frac{1}{1+x} \int_1^{+\infty} \frac{dy}{y^{1+\frac{\theta}{2}}} \frac{y-1}{y+x}] \tag{142}$$

$$C'_D / C_D = (\frac{12}{\pi} \gamma^2)^{1/3} = 1.04708 \tag{143}$$

and $\theta = (6-\epsilon)/3$ and $\zeta = \epsilon/3$. Since as $\epsilon \to 0$ for fixed x the second integral is subdominant, the leading order in ϵ are identical and the amplitude of the RSB solution compared to the FRG solution is C'_D/C_D as $\epsilon \to 0$.

4.3.3 Periodic case

In the case of a periodic system with period a, one gets a fixed point function with $\zeta = 0$ which reads:

$$\Delta_\infty(u) = \frac{\epsilon}{6} \left(\frac{a^2}{6} - u(a-u) \right) \tag{144}$$

It yields:

$$\overline{u^q u^{-q}} = a^2 c^{D/2} m^{-D} \frac{(4\pi)^{D/2}}{36 \Gamma[\frac{\epsilon}{2}]} \frac{1}{(1 + \frac{cq^2}{m^2})^{D/2}} \tag{145}$$

$$\overline{u_r^2} \sim \frac{\epsilon}{36} a^2 \ln(\frac{1}{m}) \tag{146}$$

and in $D = 4$:

$$\overline{u^q u^{-q}} = \frac{2\pi^2}{9} a^2 c^2 \frac{m^{-4}}{\ln \frac{1}{m}} \frac{1}{(1 + \frac{cq^2}{m^2})^2} \tag{147}$$

5 towards two loop FRG

The exact RG method allows to compute quantities beyond the lowest order in ϵ. It can be carried either at $T \to 0$ for fixed system size ($T = 0$ limit) or at finite T. Solving the exact RG equation at $T = 0$ requires to follow non analytic functions. This is a difficult question, e.g. distinguishing the various cumulants in the local part demands a special procedure that we have developed. This is discussed in [16, 15].

These problems do not arise at $T > 0$ where the singularity is smoothed within a boundary layer (at one loop see Section 4.2). In the Appendix F we have used the exact RG flow to third order (given in C.4) and obtain the two loop exact FRG equation for the second cumulant $R_l(u)$ at $T > 0$. At large l the effective temperature $\hat{T}_l \to 0$ and one recovers an "effective" zero temperature equation. This equation differs from the one obtained in [44] as it contains a new "anomalous" term of the form $\lambda R'''(0^+)^2 R''(u)$. We find that the coefficient of this term is universal with $\lambda = 1/2$. Interestingly this value is consistent with the renormalizability arguments given in [15]. The question of calculation of correlations and of their universality to two loops is rather delicate [16]. Let us mention here that our exact RG fixed point equation depends only on one non universal coefficient $\overline{K}^C$, which vanishes in the case of sharp cutoff. The solutions of this equation, given in Appendix F, exhibit the property of a non zero value of $R'(0^+)$, referred to as a supercusp since it is a stronger non analyticity than the one loop one ($R'''(0^+) \neq 0$). This feature is unpleasant as it naively yields (by perturbative expansion) additional divergences as $T^{-1/2}$, a sign of possible fractional dependence in ϵ (a related phenomenon is discussed in [22]). On the other hand since its magnitude is proportional to $\overline{K}^C$ it is absent in sharp cutoff calculations. This problem and its relation to the structure of the boundary layers at high orders is further examined in [16].

6 Conclusion

In this paper we have introduced a systematic method which turns the exact, though abstract, RG functional equation of Wilson-Polchinski into a tool for concrete perturbative calculations to any number of loops using arbitrary cutoff functions. The strategy was to explicitly integrate out all non local interactions, which can be expressed in terms of the local part alone, order by order in the local part. In the process we have preserved the exactness and the controlled nature of the original Wilson-Polchinski equation. Indeed, no approximation was made, and the resulting RG equation for the local part, as well as the expressions for the nonlocal ones and for the correlation functions, are formally exact order by order in an expansion in the local part. This expansion will be useful for theories where the local part is small, i.e when it is controlled by a small parameter (e.g. the shift from the upper critical dimension) and when the RG equation admits a perturbative fixed point in this parameter. We have considered here theories with a bare local interaction and a fixed point for the local part, e.g. as in the $O(n)$ model, but the method is more general and can be extended to theories where the bilocal part of the interaction serves as the small parameter, e.g. for self avoiding manifolds [34]. In a sense, the exact RG in the operational form presented here directly translates the ideas of Wilson and provides explicit checks of

universality.

In addition to presenting the method formally to all orders, we have derived the *explicit* RG equation for the local part up to third order. Further expanding in the number of loops, we have explicitly given the coefficients up to two loops and third order. Two distinct, although equivalent, methods have been presented, depending on whether one considers the Wick ordered functional or not. Each method has its advantages: the Wick ordered method yields apparently simpler (less nonlocal) RG equations, but it is not always the most adequate (e.g. for the finite T one loop FRG analysis). Although the present paper contains all the material necessary for two loop applications (e.g. for $O(n)$ and for the FRG) we have preferred to defer giving the detailed calculations and results to a companion paper [16]. In particular we have sketched here the simple extensions needed to deal with the so-called wave function renormalization which arises in e.g. the $O(n)$ model to two loops, examined in more details in [16].

We have thus considered here mostly one loop applications. The first one was a simple check to recover the one loop exponents of the $O(n)$ model. The second application was to the theory which describes elastic systems in random potentials. It was previously analyzed through simpler Wilson momentum shell integration [14] but the rather unusual nature of the theory ($n \to 0$ limit, non-analyticity) made it important to verify explicitly that the results are universal. Also universality in disordered systems is rather less established than in pure systems, especially in the $T = 0$ limit where it is known to fail in some cases. Thus we first derived the $T = 0$ one loop RG equation for the second cumulant function $R(u)$ in arbitrary cutoff scheme and found that its coefficients are universal to this order. This yields the universality to $\mathcal{O}(\epsilon)$ of the roughness exponent ζ of pinned interfaces. In the periodic case, we also explicitly verified that the correlation function contains a universal amplitude. Similarly, we computed the scaling function of the ground state deformations of a confined interface in a random field and found a universal result. This quantity can be experimentally measured in disordered magnets in the presence of a small additional field gradient.

Although temperature is formally irrelevant (the dimensionless temperature flows to zero) it is well known to be "dangerously" so. Our exact FRG at $T > 0$ shows that although the "boundary layer", i.e the detailed asymptotic form of the cumulant $R(u)$ for $u \sim T_l$ is nonuniversal, some of its features are universal, and in particular we were able to extract from it the universal divergence of the mesoscopic fluctuations of the suceptibility $\Delta\chi$. The divergence of this quantity, which is dominated by rare almost degenerate low energy configurations, is an accepted unambiguous measure of "glassiness" in a disordered system and is measured in experiments, e.g. in microsize vortex systems.

Some of the peculiar features of the theory of pinned elastic systems have been also discussed. We have found it useful to give a detailed diagrammatic proof of the triviality of naive perturbation theory, as we have not seen it explicitly in the litterature (though more general statements about dimensional reduction appear in a number of other works). We have discussed how the non analytic nature of the theory yields non trivial results.

Pinning of disordered media thus provided us here with one example of a problem where exact renormalization is needed to get insight, as no field theoretical description is yet available. The reason for it is that one must follow in principle a rather complicated object, the full probability distributions or the disorder, or equivalently the whole series of cumulants. The method seems thus promising for other problems with similar features, such as random

Sine Gordon models [35]. It is interesting to note that while presenting the FRG method, Fisher pointed out (Ref. 12 in [14]) that the momemtum-shell RG "suffers from pathologies due to the sharp cutoff", and that the cusp in $R(u)$ "requires a careful analysis of the full renormalization group". Through the use of the exact RG method presented here, we provide a simple way to integrate explicitely and exactly what is left aside in the traditional RG. We are thus able to control the approximations of former approaches and to perform new calculations. Furthermore, thanks to our general framework expressed in terms of *any* cutoff function, the universality of the results is checked.

Let us close by noting that the application of the multilocal solution to the exact RG equation seems promising also to study other disordered problems, or even give a new perspective on simpler pure problems. For instance one could apply it to wetting problems taking into account the nonlinear part, or to the roughening problems to improve on previous analysis using uncontrolled projections methods [36] The multilocal expansion allows also interesting extensions to theories with bilocal bare action, such as polymers, mutually interacting or with disorder. Finally, it is also worth studying more closely the set of exact solutions to the Polchinski equation presented in this paper (Appendix B). Some of these extensions will be explored in future publication.

A Invariance properties of generating functional and renormalization equation

In this Appendix we give a concise derivation of the exact invariance properties of the generating functional of correlation functions under coarse graining. These properties provide the basis for developing exact renormalisation procedures of the Polchinski type. In the second part we generalize the framework to include additional field transformations, such as rescaling. This extended framework is suitable for theories where wave function renormalization must be included (see [16]).

A.1 Invariance under coarse graining

We use only the two following properties of Gaussian averages. The notations are the same as in the body of the paper. First, transformation under a change of variable $\phi \to \phi + \psi$ for any field ψ in the functional integration over ϕ yields:

$$[\mathcal{A}(\phi)]_G = e^{-\frac{1}{2}\psi:G^{-1}:\psi} \left[e^{-\psi:G^{-1}:\phi} \mathcal{A}(\phi+\psi) \right]_G \tag{148}$$

for any functional $\mathcal{A}(\phi)$. We will also use the composition property:

$$\left[[\mathcal{A}(\phi_1+\phi_2)]_{G_2} \right]_{G_1} = [\mathcal{A}(\phi)]_{G_1+G_2} \tag{149}$$

where in the l.h.s. the average over ϕ_i is performed using Gaussian correlations G_i.

Using successively the shifts $\phi_1 \to \phi_1 - G_2 : J$ and $\phi_2 \to \phi_2 + G_2 : J$ yields the fundamental relation

$$\left[e^{J:\phi - \mathcal{V}(\phi)} \right]_{G_0} = \left[\left[e^{J:(\phi_1+\phi_2) - \mathcal{V}(\phi_1+\phi_2)} \right]_{G_2} \right]_{G_1} \tag{150}$$

$$= \quad e^{-\frac{1}{2}J:(G_2+G_2:G_1^{-1}:G_2):J}\left[e^{J:(1+G_2:G_1^{-1}):\phi_1}\left[e^{-\mathcal{V}(\phi_1+\phi_2)}\right]_{G_2}\right]_{G_1} \tag{151}$$

where we denoted $G_2 = G_0 - G_1$. Thus, if $\mathcal{V}_1$ is coarse–grained transformed of the interaction $\mathcal{V}$, defined by

$$e^{-\mathcal{V}_1(\phi_1)} = \left[e^{-\mathcal{V}(\phi_1+\phi_2)}\right]_{G_0-G_1} \tag{152}$$

then one has

$$\left[e^{J:\phi-\mathcal{V}(\phi)}\right]_G \quad = \quad e^{\frac{1}{2}J:(G_0-G_0:G_1^{-1}:G_0):J}\left[e^{J:G_0:G_1^{-1}:\phi_1}e^{-\mathcal{V}_1(\phi_1+\phi_2)}\right]_{G_1} \tag{153}$$

We now use this property of Gaussian integrals as follows. One defines a family of actions $\mathcal{S}_G(\phi)$ and their associated generating functional $W_G(J)$

$$\mathcal{S}_G(\phi) = \frac{1}{2}\phi : G^{-1} : \phi + \mathcal{V}_G(\phi) \qquad W_G(J) = \ln\left[e^{J:\phi-\mathcal{V}_G(\phi)}\right]_G \tag{154}$$

They are indexed by the matrix G and we choose them to be related by the coarse graining operation (152) where G plays the role of G_1, namely:

$$e^{-\mathcal{V}_G(\phi)} = \left[e^{-\mathcal{V}_{G_0}(\phi+\psi_2)}\right]_{G_0-G} \tag{155}$$

or, equivalently in a differential form, the $\mathcal{V}_G$ satisfy the "RG equation":

$$\frac{\delta}{\delta G}e^{-\mathcal{V}_G(\phi)} = \frac{1}{2}\frac{\delta^2}{\delta\phi\delta\phi}e^{-\mathcal{V}_G(\phi)} \tag{156}$$

obtained by differentiating (155) with respect to G. The coarse–graining equation (156), read along a given path $l \mapsto G_l$, is the Polchinski equation in its "diffusive" form (14).

It is easy to see from (153) that this choice of a family $\mathcal{V}_G$ implies the property:

$$\tilde{W}_G(J : G' : G^{-1}) \text{ independent of } G \tag{157}$$

where we have defined the interaction part $\tilde{W}_G(J) = W_G(J) - \frac{1}{2}J : G : J$. It allows to relate correlations within any member of the family $\mathcal{S}_G$, i.e under coarse graining.

A.2 Generalization including rescaling and change in Gaussian part

The previous properties can be extended to a larger set of transformations which include simultaneous (i) coarse graining (ii) linear transformation of the fields (iii) redefinition of the Gaussian part (such as needed to absorb its possible renormalization). It is based on the following properties of Gaussian integrals. One defines:

$$W_{G,\mathcal{V}}(J) = \ln[e^{J:\phi-\mathcal{V}(\phi)}]_G \tag{158}$$

The first property correspond to performing an arbitrary linear transformation on the field:

$$W_{G,\mathcal{V}}(J) = W_{M^{-1}:G:M^{-1},\mathcal{V}(M:\phi)}(J : M) \tag{159}$$

valid for any $G, \mathcal{V}, J, M$. The second property is simply the identity obtained when redistributing the Gaussian part:

$$W_{G,\mathcal{V}}(J) = -\frac{1}{2}\mathrm{Tr}\ln(1+H:G) + W_{(G^{-1}+H)^{-1},\mathcal{V}(\phi)-\frac{1}{2}\phi:H:\phi}(J) \tag{160}$$

valid for any $G, \mathcal{V}, J, H$.

First, let $\mathcal{V}_G(\phi)$ satisfy the RG equation (156). Then from the previous Section we know that $W_{G,\mathcal{V}_G}(J:G':G^{-1}) - \frac{1}{2}J:G':G^{-1}:G':J$ is independent of G for any J, G'. Setting $G = G_l$ and $G' = G_0$ one gets the Polchinski equation and one can compute $W(J)$. One now defines

$$\mathcal{V}_{G,M}(\phi) = \mathcal{V}_G(M:\phi) + \frac{1}{2}\phi:(M:G^{-1}:M-G^{-1}):\phi \tag{161}$$

Using the above properties one has

$$W_{G,\mathcal{V}_{G,M}}(J:G':G^{-1}:M) - \frac{1}{2}J:G':G^{-1}:G':J - \frac{1}{2}\mathrm{Tr}\ln(M^{-1}:G^{-1}:M^{-1}:G) \tag{162}$$

is independent of G and M, for any G', J.

We have used the two invariances choosing $H = M^{-1}:G^{-1}:M^{-1}-G^{-1}$ leading to the intermediate formula:

$$W_{G,\mathcal{V}}(J) = -\frac{1}{2}\mathrm{Tr}\ln(M^{-1}:G^{-1}:M^{-1}:G) + W_{G,\mathcal{V}(M:\phi)+\frac{1}{2}\phi:(M:G^{-1}:M-G^{-1}):\phi}(J:M) \tag{163}$$

Defining a new family of functional indexed by l as:

$$\mathcal{V}_l = \mathcal{V}_{G_l,M_l} \tag{164}$$

and symmetric matrices G, M one finds that the functional $\mathcal{V}_l$ now satisfies a new RG equation

$$\partial_l\mathcal{V} = \frac{\partial\mathcal{V}}{\partial\phi}:M^{-1}:\partial M:\phi + \phi:G^{-1}:M^{-1}:\partial M:\phi \tag{165}$$

$$-\frac{1}{2}\mathrm{Tr}(\partial G:\frac{\partial^2\mathcal{V}}{\partial\phi\partial\phi}) + \frac{1}{2}\frac{\partial\mathcal{V}}{\partial\phi}:\partial G:\frac{\partial\mathcal{V}}{\partial\phi} \tag{166}$$

which contains additional terms. In particular the quadratic piece can be used to absorb "wave function renormalization" terms, so as to keep $\mathcal{V}_l$ small. Once this equation is solved the correlations can be related within any of the corresponding $\mathcal{S}_l$ theories using the above invariance property (162) of $W_{G,\mathcal{V}_{G,M}}(J)$. This will be further exploited in [16].

B General properties and exact solutions of Polchinski equation

Let us first mention a few general properties of (9,14). For the class of cutoff functions (5) used in practice, the diffusion tensor in (14) is positive $c'(s) \leq 0$ (but not definite

positive since there exists modes with $\partial_l G_l^q = 0$). There are some exactly formally conserved quantities, such as $\int_\phi e^{-\mathcal{V}_l(\phi)}$ and $\left[e^{-\mathcal{V}_l(\phi)}\right]_{G_l}$. Since (14) is a diffusion equation, it satisfies a H-theorem of increase of the "entropy" $\mathsf{S}_l = \int_\phi \mathcal{V}_l(\phi) e^{-\mathcal{V}_l(\phi)}$, which flows as $\partial_l \mathsf{S}_l = -\frac{1}{2}\int_\phi \frac{\delta}{\delta\phi} e^{-\mathcal{V}_l(\phi)} : \partial_l G_l : \frac{\delta}{\delta\phi} e^{-\mathcal{V}_l(\phi)} \geq 0$ and is compatible with the fact that RG trajectories do not have limit cycles. Finally, since (14) is a linear equation, if we now a set of solutions $\mathcal{V}_l^\alpha(\phi)$, then any superposition such as

$$\mathcal{V}_l(\phi) = -\ln \sum_\alpha c^\alpha e^{-\mathcal{V}_l^\alpha(\phi)} \tag{167}$$

is also solution.

This can now be used to construct non trivial exact solutions to the Polchinski equation. The simplest family of exact solutions is of course the quadratic potential, for which one finds the solutions

$$\mathcal{V}_l(\phi) = \frac{1}{2}(\phi - \psi) : M_l : (\phi - \psi) - \frac{1}{2}\mathrm{Tr}\ln M_l \tag{168}$$

$$M_l = \left(M_0^{-1} + G_0 - G_l\right)^{-1} \tag{169}$$

where ψ is an l-independent field, with $\langle\phi\rangle_{\mathcal{S}} = (1 + M^{-1} : G^{-1})^{-1} : \psi$.

A much less trivial family of *exact solutions* of Polchinski equation is obtained by superposition of gaussians, i.e of quadratic potentials. It reads:

$$\mathcal{V}_l(\phi) = -\ln \sum_\alpha c^\alpha \ e^{-\frac{1}{2}(\phi-\psi^\alpha):M_l^\alpha:(\phi-\psi^\alpha) + \frac{1}{2}\mathrm{Tr}\ln M_l^\alpha} \tag{170}$$

with arbitrary constant coefficients c^α and each M_l^α satisfies (168). This is somewhat reminiscent of a decomposition into "pure states" and is clearly of interest to describe low temperature states in pure models (in phases with broken symmetry) or in disordered models and glasses (with many metastable states). It is an interesting question to ask, quite generally, whether this family of solutions can in some cases be an *attractive* manifold in a larger functional space, or whether one can carry perturbation around this subspace. These and related issues will be discussed in a future publication [45]

A generic property of these solutions to Polchinski equation is to generate cusp singularities separating the "pure states". This can be seen directly above since negative curvatures tend to increase in absolute value (see (168) and is presumably a very general mechanism. It can also be seen on the simple example of the zero-dimensional toy model. There, the field ϕ is a real number and

$$Z = \int_{-\infty}^{\infty} d\phi\, e^{-\frac{\phi^2}{2G} - V(\phi)} \tag{171}$$

where V is an arbitrary function. One can introduce $G_l = G - l$, and $V_l(\phi)$ for this model, verifying

$$\partial_l V_l(\phi) = \frac{1}{2}\left(V_l''(\phi) - V_l'(\phi)^2\right) \tag{172}$$

with initial condition $V_0(\phi) = V(\phi)$, and one can integrate up to $l = G$. One has $e^{-V_l(\phi)} = \left[e^{-V(\phi+\psi)}\right]_l$. The evolution of $V_l(\phi)$ is that the curvature $M_l = V_l''(0)$, which would obey

(168) $M_l = M_0/(1+M_0 l)$ for a quadratic hill or well, diverges at a finite l for maxima and decreases as $1/l$ for minima of $V(\phi)$. Thus the landscape $V_l(\phi)$ develops cusps, encoding for discontinuities in the force $-V'(\phi)$.

In the case of a periodic landscape, the natural superposition of gaussian solutions is the *Villain potential* $V(\phi) = -\ln \sum_n c e^{-(\phi-n)^2/(2l)}$. In these Sine gordon type potential, as well as in the $2D$ XY model, it is a well known property that the renormalized potential converges towards the Villain form at low temperature as found in [37] from the Migdal Kadanoff RG (see also more recently [36]). The detailed behaviour of the RG flow can be studied in a more controlled way using the method presented of this paper [45]

C Multilocal expansion and higher order RG equation

In this Appendix we derive the systematic multilocal expansion and obtain the RG equation to higher orders. We give a detailed presentation for the functional $\hat{\mathcal{V}}_l(\phi)$, which is simpler, and give explicitly the corresponding RG equation to order $\hat{\mathcal{V}}_l^3$ and up to two loops. Then we simply sketch the result for the same procedure applied to the functional $\mathcal{V}_l(\phi)$, which is more involved and will be presented in [16].

C.1 Multilocal expansion for $\hat{\mathcal{V}}$

The tadpole-free functional $\hat{\mathcal{V}}_l(\phi)$, defined in (42), can be written as a sum of multilocal interactions

$$\hat{\mathcal{V}}_l(\phi) = \sum_{p>0} \int_{x_1..x_p} V^{(p)}(\phi_{x_1}..\phi_{x_p}, x_1..x_p) \tag{173}$$

Note that we are not even assuming here translational invariance. The translationally invariant case discussed in (2.2) can be recovered by setting $V_l^{(1)}(\phi_1, x_1) = \hat{U}_l(\phi_1)$, $V_l^{(2)}(\phi_1, \phi_2, x_1, x_2) = \hat{V}_l(\phi_1, \phi_2, x_1 - x_2)$ etc.. Since we want to impose that each $V^{(p)}$, $p > 1$, has zero local part (this is sufficient for our purpose), we define (extending (19,20)) respectively the projection operator $\overline{P_1}$ which projects a p-local interaction on a local one, and the projection operator P_1 which transform a p-local interaction into another p-local interaction as:

$$(\overline{P}_1 A)(\phi, t) = \int_{x_1..x_p} \delta(t - \frac{x_1 + \ldots + x_p}{p}) A(\phi, ..\phi, x_1, ..x_p) \tag{174}$$

$$(P_1 A)(\phi_1, ..\phi_p, x_1, ..x_p) = \delta(x_1 - x_2) \ldots \delta(x_1 - x_p) \tag{175}$$
$$\times \int_{y_1 \ldots y_p} \delta(x_1 - \frac{y_1 + \ldots + y_p}{p}) A(\phi_1 \ldots \phi_p, y_1 \ldots y_p)$$

The property

$$\int_{x_1,..x_p} (P_1 A)(\phi_{x_1}, ..\phi_{x_p}, x_1, ..x_p) = \int_t (\overline{P}_1 A)(\phi_t, t) = \int_t A(\phi_t, ..\phi_t, t, ..t) \tag{176}$$

ensures that one can choose the $V^{(p)}$, $p > 1$ in the decomposition (173) to have no local part, i.e:

$$P_1 V_l^{(p)} = 0 \qquad \overline{P}_1 V_l^{(p)} = 0 \tag{177}$$

for any l by applying P_1 and $1-P_1$ act on both sides of the Polchinski equation.

Since the modified Polchinski equation (43) concatenates two operators, it is then easy to see that if the $V_l^{(p)}$ satisfy the following set of equations:

$$\partial_l V^{(1)}(\phi,t) = \frac{1}{2}\sum_{p>0}\sum_{q=1}^{p-1}\int_{x_1\dots x_p}\delta(t-\frac{x_1+\ldots+x_p}{p})e^{\partial^{1\dots q}G_l\partial^{q+1\dots p}}\partial^{1\dots q}\partial G_l\partial^{q+1\dots p}$$
$$V^{(q)}(\phi_{x_1}\dots\phi_{x_q},x_1\dots x_q)V^{(p-q)}(\phi_{x_{q+1}}\dots\phi_{x_p},x_{q+1}\dots x_p)|_{\phi_i=\phi} \quad (178)$$

$$\partial_l V^{(p)}(\phi_{x_1}\dots\phi_{x_q},x_1\dots x_q) = \frac{1}{2}\mathbf{S}(1-P_1)\sum_{q=1}^{p-1}e^{\partial^{1\dots q}G_l\partial^{q+1\dots p}}\partial^{1\dots q}\partial G_l\partial^{q+1\dots p}$$
$$V^{(q)}(\phi_{x_1}\dots\phi_{x_q},x_1\dots x_q)V^{(p-q)}(\phi_{x_{q+1}}\dots\phi_{x_p},x_{q+1}\dots x_p) \text{ for } p>1 \quad (179)$$

then (43) is obeyed by $\hat{\mathcal{V}}_l(\phi)$. Since we prefer to work with symmetric functions we have defined the symmetrization operator:

$$\mathbf{S}B^{(p)}(\phi_1\dots\phi_p,x_1\dots x_p) = \frac{1}{p!}\sum_{\sigma\in\Sigma_p}B^{(p)}(\phi_{\sigma(1)}\dots\phi_{\sigma(p)},x_{\sigma(1)}\dots x_{\sigma(1)}) \quad (180)$$

we have also defined the following shorthand notations:

$$\partial^{1\dots q}\partial G\partial^{q+1\dots p} = \sum_{\alpha=1}^{q}\sum_{\beta=q+1}^{p}\partial G_{ij}^{x_\alpha x_\beta}\partial_i^\alpha\partial_j^\beta \quad (181)$$

It is easy to see that if $V^{(1)}$ is considered formally as "small" in some sense (e.g. controlled by a small parameter such as ϵ) then one can integrate exactly these equations order by order in $V^{(1)}$ and check that $V^{(p)} = \mathcal{O}(V^{(1)p})$. More precisely, to a given order one can exactly integrate the equations for higher point functions and reduce to a single equation for $V^{(1)}$. This is the procedure that we now follow. The structure to the lowest order $O({V^{(1)}}^2)$ is simply a closed equation for $V^{(1)}$ of the schematic form:

$$\partial_l V^{(1)} = \overline{P_1}(V^{(1)} * V^{(1)}) + O({V^{(1)}}^3) \quad (182)$$

To next order $O({V^{(1)}}^3)$ one needs to solve the coupled set:

$$\partial_l V^{(1)} = \overline{P_1}(V^{(1)} * V^{(1)} + V^{(1)} * V^{(2)}) + O({V^{(1)}}^4) \quad (183)$$
$$\partial_l V^{(2)} = (1-P_1)(V^{(1)} * V^{(1)}) \quad (184)$$

The second equation is explicitly integrated which yields $V^{(2)}[V^{(1)}]$ which is then substituted in the first equation, producing a closed equation for $V^{(1)}$. This procedure can be extended to any order in $V^{(1)}$. We now give the explicit calculation.

C.2 RG equation up to order ${V^{(1)}}^3$

To (lowest) order ${V^{(1)}}^2$, the beta function is local in l as the modified Polchinski equation itself and reads

$$\partial_l V^{(1)}(\phi,t) = \frac{1}{2}\int_{x_1x_2}\delta(t-\frac{x_1+x_2}{2})e^{\partial^1 G_l\partial^2}\partial^1\partial G_l\partial^2 V^{(1)}(\phi_1,x_1)|_{\phi_1=\phi}V^{(1)}(\phi_2,x_2)|_{\phi_2=\phi} \quad (185)$$

up to terms of order $\mathcal{O}({V^{(1)}}^3)$.

To next order ${V^{(1)}}^3$, as explained above one first compute the bi-local operator as a function of $V^{(1)}$. Its flow equation to the necessary order reads:

$$\begin{aligned}\partial_l V_l^{(2)}(\phi_1\phi_2, x_1x_2) &= \frac{1}{2}e^{\partial^1 G_l\partial^2}\partial^1\partial G_l\partial^2 V^{(1)}(\phi_1,x_1)V^{(1)}(\phi_2,x_2) \\ &\quad -\delta(x_1-x_2)\int_{y_1y_2}\delta(x_1-\frac{y_1+y_2}{2})e^{\partial^1 G_l\partial^2}\partial^1\partial G_l\partial^2 V^{(1)}(\phi_1,y_1)V^{(1)}(\phi_2,y_2)\end{aligned} \tag{186}$$

up to $\mathcal{O}({V^{(1)}}^3)$ terms. Integrating $\partial_\mu V_\mu^{(2)}(\phi_1\phi_2, x_1x_2)$ using (186) from 0 to l and substituting the result into the equation for $V^{(1)}$ one finds the RG equation of the local part of the interaction to order ${V^{(1)}}^3$:

$$\partial_l V^{(1)}(\phi,t) = \frac{1}{2}\int_{x_1x_2}\delta(t-\frac{x_1+x_2}{2})e^{\partial^1 G_l\partial^2}\partial^1\partial G_l\partial^2 V^{(1)}(\phi_1,x_1)V^{(1)}(\phi_2,x_2) \tag{187}$$

$$\begin{aligned}&+\frac{1}{2}\int_{x_1x_2x_3}\delta(t-\frac{x_1+x_2+x_3}{3})e^{\partial^{12} G_l\partial^3}\partial^{12}\partial G_l\partial^3\Big(\int_0^l d\mu \\ &\Big[e^{\partial^1 G_\mu\partial^2}\partial^1\partial G_\mu\partial^2 V_\mu^{(1)}(\phi_1,x_1)|_{\phi_1=\phi}V_\mu^{(1)}(\phi_2,x_2)|_{\phi_2=\phi} - \delta(x_1-x_2)\int_{y_1y_2}\delta(x_1-\frac{y_1+y_2}{2}) \\ &e^{\partial^1 G_\mu\partial^2}\partial^1\partial G_\mu\partial^2 V_\mu^{(1)}(\phi_1,y_1)|_{\phi_1=\phi}V_\mu^{(1)}(\phi_2,y_2)|_{\phi_2=\phi}\Big]V_l^{(1)}(\phi_3,x_3)|_{\phi_3=\phi}\Big)\end{aligned} \tag{188}$$

up to $\mathcal{O}({V^{(1)}}^4)$ terms.

C.3 Translation invariant theory and loop expansion

In a spatially translational invariant theory the local interaction does not depend explicitly on the space variable t, $V^{(1)}(\phi,t) = \hat{U}_l(\phi)$. The above formulas, when expanding the exponentials in a loop expansion, possess a representation in terms of Feynman graphs as indicated in Fig. 3. Interestingly, all one particle reducible graphs vanish due to the property $\partial_l G_l^{q=0} = 0$ ($c'(0) = 0$). In addition, since each graph to order $\hat{U}^3$ possesses a counterpart with a minus sign which is the product of two (factorized) graphs with independent sets of loop integrations, this automatically cancels all such (factorized) graphs.

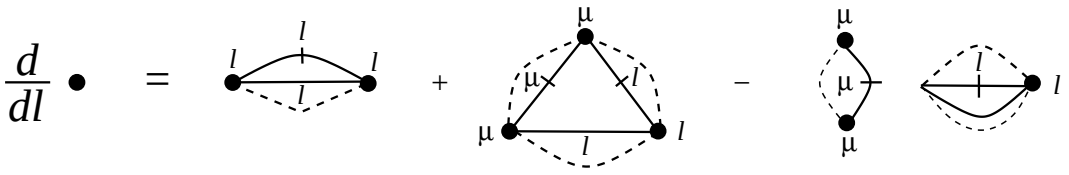

Figure 3: Graphical representation of the expansion of the modified RG equation in powers of the local part up to $\mathcal{O}({V^{(1)}}^3)$. The points represent the vertices $\hat{U}_l$, the broken solid lines are propagators on shell ∂G, the full solid lines are propagators G, the indices l or μ are indicated at each vertex and at each bond. While the solid lines are necessarily present, additional dashed lines appear in arbitrary number when performing the loop expansion.

The RG equation, at any given order in $\hat{U}_l$, can be further expanded in the number of loops by expanding the exponentials in (187). Let us give the specific result for the case of

a diagonal gaussian part $G^q_{l,ij} = \delta_{ij} G^q_l$, the generalization being straighforward. To order $\hat{U}^3_l$ and up to two loops we obtain from (187) the RG equation for $\hat{U}_l(\phi)$ as:

$$\partial_l \hat{U}_l(\phi) = I^D_l D_l(\phi) + I^F_l F_l(\phi) + \int_0^l d\mu \left(I^T_{l\mu} T_{l\mu}(\phi) + I^A_{l\mu} A_{l\mu}(\phi) + I^{A'}_{l\mu} A'_{l\mu}(\phi) \right) \quad (189)$$

up to $O(\hat{U}^4_l)$ terms, where the contraction graphs are

$$D_l(\phi) = \partial_{ij} \hat{U}_l(\phi) \partial_{ij} \hat{U}_l(\phi) \quad (190)$$

$$F_l(\phi) = \partial_{ijm} \hat{U}_l(\phi) \partial_{ijm} \hat{U}_l(\phi) \quad (191)$$

$$T_{l\mu}(\phi) = \partial_{ij} \hat{U}_l(\phi) \partial_{jm} \hat{U}_\mu(\phi) \partial_{mi} \hat{U}_\mu(\phi) \quad (192)$$

$$A_{l\mu}(\phi) = \partial_{ij} \hat{U}_l(\phi) \partial_{imn} \hat{U}_\mu(\phi) \partial_{jmn} \hat{U}_\mu(\phi) \quad (193)$$

$$A'_{l\mu}(\phi) = \partial_{ij} \hat{U}_\mu(\phi) \partial_{imn} \hat{U}_\mu(\phi) \partial_{jmn} \hat{U}_l(\phi) \quad (194)$$

and the momenta graphs read

$$I^D_l = \frac{1}{2} \int_q \partial G^q_l G^q_l \quad (195)$$

$$I^F_l = \frac{1}{4} \int_{q_1 q_2 q_3} \delta_{q_1+q_2+q_3} \partial G^{q_1}_l G^{q_2}_l G^{q_3}_l \quad (196)$$

$$I^T_{l\mu} = \int_q G^q_l \partial G^q_l \partial G^q_\mu \quad (197)$$

$$I^A_{l\mu} = \int_{q_1 q_2 q_3} (\delta_{q_1+q_2+q_3} - \delta_{q_2+q_3}) G^{q_1}_l \partial G^{q_1}_l G^{q_2}_\mu \partial G^{q_3}_\mu \quad (198)$$

$$I^{A'}_{l\mu} = \int_{q_1 q_2 q_3} \delta_{q_1+q_2+q_3} (\frac{1}{2} \partial G^{q_1}_l G^{q_2}_l G^{q_3}_l \partial G^{q_1}_\mu + \partial G^{q_2}_l G^{q_3}_l G^{q_1}_l \partial G^{q_1}_\mu) \quad (199)$$

Note that to two loops the RG flow is generically non local in l. The values of the above integrals will be computed in [16].

C.4 RG equation for $U_l(\phi)$

The systematic expansion in multilocal interactions can also be performed directly on the functional $\mathcal{V}((\phi)$. The procedure parallels the previous section and its details are given in [16]. Here we give only the result for a translationally invariant theory, for the RG flow of $U_l(\phi)$ in the translationally invariant case to order U^3_l. It reads:

$$\partial U_l(\phi) = -\frac{1}{2} (\partial G_l)^{x=0}_{\alpha\beta} \partial_\alpha \partial_\beta U_l(\phi) \quad (200)$$

$$-\frac{1}{4} \int_{x_1 x_2} \delta(\frac{x_1+x_2}{2}) \partial^{12} (\partial G_l - \partial G^{x=0}_l) \partial^{12} \int_0^l d\mu \, e^{-\frac{1}{2} \partial^{12} G_{l\mu} \partial^{12}} \partial^1 \partial G_\mu \partial^2 U_\mu(\phi_1)_{|\phi_1=\phi} U_\mu(\phi_2)_{|\phi_2=\phi}$$

$$-\frac{1}{4} \int_{x_1 x_2 x_3} \delta(\frac{x_1+x_2+x_3}{3}) \int_0^l d\mu$$

$$\left[\partial^{123} (\partial G_l - \partial G^{x=0}_l) \partial^{123} e^{-\frac{1}{2} \partial^{123} G_{l\mu} \partial^{123}} \partial^{12} \partial G_\mu \partial^3 - \text{idem } x_1 = x_2 \equiv \frac{x_1+x_2}{2} \right]$$

$$\int_0^\mu d\nu \, e^{-\frac{1}{2} \partial^{12} G_{\mu\nu} \partial^{12}} \partial^1 \partial G_\nu \partial^2 U_\nu(\phi_1)_{|\phi_1=\phi} U_\nu(\phi_2)_{|\phi_2=\phi} U_\mu(\phi_3)_{|\phi_3=\phi}$$

Note that the idem $x_1 = x_2 \equiv \frac{x_1+x_2}{2}$ applies only in the square bracket to the arguments of the G's. We have also defined:

$$\partial^{1\dots p}\partial G\partial^{1\dots p} = \sum_{\alpha,\beta=1}^{p} \partial G_{ij}^{x_\alpha x_\beta}\partial_i^\alpha\partial_j^\beta \tag{201}$$

Again the 1PR diagrams are elimnated by construction since they have one point x_i on which one can integrate freely, producing a $\partial G^{q=0}$ which vanishes by construction. The loop expansion of this formula will be detailed in [16].

C.5 Computation of the correlation functions via RG

The invariance of the generating functional $W(J)$ of the (connected) correlation functions with respect to l is now used as a tool for computing the correlation functions of the initial model $\mathcal{S}_0$. The expansion of $W(J)$ in powers of the running interaction $\mathcal{V}_l(\phi)$ reads formally to all orders:

$$W(J) = \frac{1}{2}J:G:J + \sum_{m=1}^{+\infty}\kappa_m \tag{202}$$

where we have defined:

$$\mu_n = \left[e^{J:G:G_l^{-1}:\phi}(\mathcal{V}_l(\phi))^n\right]_{G_l} \tag{203}$$

$$Y = e^{-\frac{1}{2}J:G:G_l^{-1}:G:J} \tag{204}$$

$$\sum_{m=1}^{+\infty}\kappa_m x^m = \ln(1+Y\sum_{n=1}^{+\infty}\frac{(-x)^n}{n!}\mu_n) \tag{205}$$

Up to second order, this expansion reduces to:

$$W(J) = \frac{1}{2}J:G:J - e^{-\frac{1}{2}J:G:G_l^{-1}:G:J}\left[e^{J:G:G_l^{-1}:\phi}\mathcal{V}_l(\phi)\right]_{G_l} \tag{206}$$
$$+\frac{1}{2}e^{-\frac{1}{2}J:G:G_l^{-1}:G:J}\left[e^{J:G:G_l^{-1}:\phi}\mathcal{V}_l(\phi)^2\right]_{G_l} - \frac{1}{2}(e^{-\frac{1}{2}J:G:G_l^{-1}:G:J}\left[e^{J:G:G_l^{-1}:\phi}\mathcal{V}_l(\phi)\right]_{G_l})^2 + O(\mathcal{V}_l^3)$$

The interaction functional $\hat{\mathcal{V}}_l(\phi)$ defined by (42) naturally appears in the expansion of $W(J)$. Using the properties (148) and $[\mathcal{A}(\phi)]_G = e^{\frac{1}{2}\frac{\delta}{\delta\phi}:G:\frac{\delta}{\delta\phi}}\mathcal{A}(0)$, one obtains:

$$W(J) \;=\; \frac{1}{2}J:G:J - \hat{\mathcal{V}}_l(G:J) + \frac{1}{2}\left(\widehat{\mathcal{V}_l^2}(G:J) - \hat{\mathcal{V}}_l^2(G:J)\right) + \mathcal{O}(\mathcal{V}_l^3) \tag{207}$$

where $\widehat{\mathcal{V}_l^2}(\psi) = e^{\frac{\delta}{\delta\phi_1}:G_l:\frac{\delta}{\delta\phi_2}}\hat{\mathcal{V}}_l(\phi_1)\hat{\mathcal{V}}_l(\phi_2)|_{\phi_1=\phi_2=\psi}$. On this expression, it becomes obvious that $W(J)$ is indeed l-independent (order by order), as a consequence of the RG equation for $\hat{\mathcal{V}}_l(\phi)$. As is clear from these formulae, all external legs of correlation functions will carry the propagator G while all internal legs will carry G_l.

We must now distinguish between the two methods which consist in performing the multi-local expansion on $\mathcal{V}_l(\phi)$, $\hat{\mathcal{V}}_l(\phi)$ respectively. Before doing so, we give a formula, in Fourier representation, which is valid in both cases:

$$W(J) = \frac{1}{2} J : G : J - \int_{xK} \hat{U}_l^K e^{iK.(G:J)^x} \tag{208}$$

$$- \int_{xyKP} e^{iK.(G:J)^x + iP.(G:J)^y} \left(\hat{V}_l^{KP}(x-y) - \frac{1}{2}(e^{-K.G_l^{xy}.P} - 1)\hat{U}_l^K \hat{U}_l^P \right) \tag{209}$$

$$\hat{U}_l^K = U_l^K e^{-\frac{1}{2}K.G_l^{x=0}.K} \tag{210}$$

$$\hat{V}_l^{KP}(x) = V_l^{K,P}(x) e^{-\frac{1}{2}K.G_l^{x=0}.K - \frac{1}{2}P.G_l^{x=0}.P - K.G_l^x.P} \tag{211}$$

the way to compute the functions $\hat{U}_l$ and $\hat{V}_l$ being however different in each case. Inserting the corresponding formula for $\hat{V}_l$ as a function of $\hat{U}_l$ yields expressions in terms of $\hat{U}_l$ only, which we now give in each case (for variety, we also alternate between the -equivalent - field and Fourier representations).

C.5.1 Method with $\hat{\mathcal{V}}_l(\phi)$

We start with the formalism using the multi-local expansion of $\hat{\mathcal{V}}_l$. One finds:

$$W(J) = \frac{1}{2} J : G : J - \int_x \hat{U}_l((G:J)^x) + \frac{1}{2} \int_{xy} \int_{l'>0} \; [\delta_{ll'}(e^{\partial_1 G_l^{x-y} \partial_2} - 1) \tag{212}$$

$$-\theta_{ll'} \partial_{l'} (e^{\partial^1 G_{l'}^{x-y} \partial^2} - \delta(x-y) \int_z e^{\partial_1 G_{l'}^z \partial_2}) \hat{U}_{l'}(\phi_1)|_{\phi_1 = (G:J)^x} \hat{U}_{l'}(\phi_2)|_{\phi_2 = (G:J)^y}] \tag{213}$$

up to $O(U^3)$ terms. We denote $\delta_{ll'} = \delta(l-l')$ and $\theta_{ll'} = \theta(l-l')$. From this formula one can compute all connected correlations to $O(U^2)$. Let us give the self energy, defined as usual from the two point function $C = G + \delta C$ as:

$$\Sigma = C^{-1} - G^{-1} = -G^{-1} \delta C G^{-1} + G^{-1} \delta C G^{-1} \delta C G^{-1} + O(\delta C^3) \tag{214}$$

It reads:

$$\Sigma_{ij}^{q=0} = \partial_i \partial_j \hat{U}_l(0) - \int_x [(\partial_i^1 \partial_j^1 + \partial_i^1 \partial_j^2)(e^{\partial^1 G_l^x \partial^2} - 1) - \partial_i^1 \partial_j^2 \partial^1 G^x \partial^2] \hat{U}_l(\phi_1) \hat{U}_l(\phi_2)|_{\phi_i = 0}$$

$$\Sigma^q - \Sigma^{q=0} = \int_x (e^{iqx} - 1) \Sigma^x \tag{215}$$

$$\Sigma_{ij}^x = -\partial_i^1 \partial_j^2 \int_{l'>0} [\delta_{ll'}(e^{\partial^1 G_l^x \partial^2} - \partial^1 G^x \partial^2 - 1) - \theta_{ll'} \partial^1 \partial G_{l'}^x \partial^2 e^{\partial^1 G_{l'}^x \partial^2}] \hat{U}_{l'}(\phi_1) \hat{U}_{l'}(\phi_2)|_{\phi_i = 0}$$

Note that it involves a term with a G propagator.

C.5.2 Method with $\mathcal{V}_l(\phi)$

Inserting the multilocal expansion of $\mathcal{V}_l$, (206) transforms into an expansion in powers of the local interaction $U_l(\phi)$:

$$W(J) = \frac{1}{2} J : G : J - \int_x \int_K \hat{U}_l^K e^{iK.G_x : J} + \frac{1}{2} \int_{xy} \int_{KP} e^{iK.G_x : J + iP.G_y : J} [\hat{U}_l^K \hat{U}_l^P (e^{-K.G_l^{x-y}.P} - 1)$$

$$- \int_0^l dl' \hat{U}_{l'}^K \hat{U}_{l'}^P \partial_{l'} \left(e^{-K.G_{l'}^{x-y}.P} - \delta(x-y) \int_z e^{-K.G_{l'}^z.P + K.(G_l^z - G_l^0).P} \right)] + O(U^3) \tag{216}$$

Using the RG equation for U_l, it is easily checked again that this expression is l-independent order by order. From (212), one can compute the self energy Σ^q of the theory. One gets to order U_l^2:

$$\Sigma_{ij}^{q=0} = -\int_K K_i K_j U_l^K e^{-\frac{1}{2}KG_l^{x=0}K} + \int_{KP}(K_i K_j A_l^{KP}(q=0) + K_i P_j B_l^{KP}(q=0)) \quad (217)$$
$$\Sigma^q - \Sigma^{q=0} = \int_{KP} K_i P_j (B_l^{KP}(q) - B_l^{KP}(q=0))$$

where

$$A_l^{KP}(q) = \int_{l'>0} U_{l'}^K U_{l'}^P e^{-\frac{1}{2}KG_{l'}^{x=0}K + PG_{l'}^{x=0}P}(\delta_{l-l'}\int_x e^{iqx}(e^{-KPG_l^x} - 1) \quad (218)$$
$$B_l^{KP}(q) = \int_{l'>0} U_{l'}^K U_{l'}^P e^{-\frac{1}{2}KG_{l'}^{x=0}K + PG_{l'}^{x=0}P}(\delta_{l-l'}\int_x e^{iqx}(e^{-KG_l^x P} + KG_0^x P - 1) \quad (219)$$
$$+\theta_{l-l'}\int_x (e^{iqx} - e^{K.(G_l^x - G_l^0).P})K\partial G_{l'}^x P e^{-KG_{l'}^x P})$$

D Dimensional reduction from graphs

D.1 Perturbation theory

In this appendix, we sketch diagrammatically how the perturbation expansion in R of the average of any observable $A[u]$ at $T=0$ is the same as the one which would be obtained in the Gaussian theory corresponding to a simple random force.

Precisely, the actions

$$\mathcal{S}[u] = \frac{1}{2T}\sum_a \int_{xy} u_a^x (\overline{G}^{-1})^{xy} u_a^y - \frac{1}{2T^2}\sum_{ab}\int_x R(u_a^x - u_b^x) \quad (220)$$

and

$$\mathcal{S}_{\rm rf}[u] = \frac{1}{-2R''(0)}\sum_{ab}\int_{xy} u_a^x ((\overline{G} * \overline{G})^{-1})^{xy} u_b^y \quad (221)$$

where $(\overline{G} * \overline{G})^x = \int_z \overline{G}^{x-z}\overline{G}^z$ yield the same results when computing the average of any functional $A[u]$ of the replicated field at $T=0$, e.g. $A[u] = \prod_i u_{a_i}^{x_i}$.

To this aim, one first show that the perturbation expansion within (220) is well-defined at $T=0$. We use a diagrammatics with propagator

$$\langle u_a^x u_b^y \rangle = T\delta_{ab}\overline{G}^{xy} \quad (222)$$

which conserves the replica index, and vertex

$$-\frac{1}{2T^2}\sum_{ab}\int_x R(u_a^x - u_b^x) \quad (223)$$

associated to *one* point in space but involving a summation over *two* replica indices. Thus we choose to split the vertex into two subvertices corresponding to each replica index.

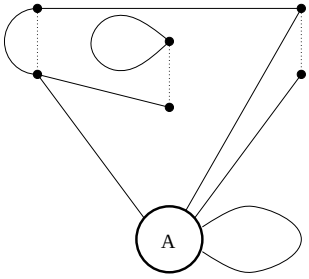

Figure 4: Typical graph contributing to the computation of the average of an observable A to third order ($p = 3$) in perturbation theory. Note here that $v = 2p = 6$, $t = 1$, $k = 3$, $e = 4$, $c = 3$, $l = 1$.

For any graph occuring in the computation of $\langle A[u]R^p\rangle^c_{\mathcal{S}}$, let us denote by k the number of lines connecting A to the vertices (involving the extraction of k legs from A : $\partial_{u^{x_1}_{a_1}\dots u^{x_k}_{a_k}} A[u]$). Let $\mathcal{K}$ be the graph obtained by considering only the splitted vertices, the propagators between them, forgetting the observable and the k lines attached to it. The graph $\mathcal{K}$ has $v = 2p$ subvertices. Contrarily to the initial graph with unsplitted vertices, $\mathcal{K}$ is not necessarily connected and is made of c connected components. To each one corresponds a replica index. If one of them is not connected to the observable, i.e. if it does not inherit from a replica index contained in A, then the summation over this index is free, giving a factor n. Hence each connected component has to be linked to the observable in order to survive the $n \to 0$ limit, which yields $k \geq c$.

Collecting the factors of T in front of the initial graph ($\mathcal{K}$, A, the k propagators, and possibly t tadpoles on A), the power of T is $t+e+k-v$ where e is the number of propagators in $\mathcal{K}$. Euler relation in $\mathcal{K}$ reads $v + l = c + e$ where l is the number of loops in $\mathcal{K}$. But since $k \geq c$ and $l \geq 0, t \geq 0$, one obtains that each graph is in factor of a non-negative power of T. The existence of the $T = 0$ perturbation theory is thus confirmed.

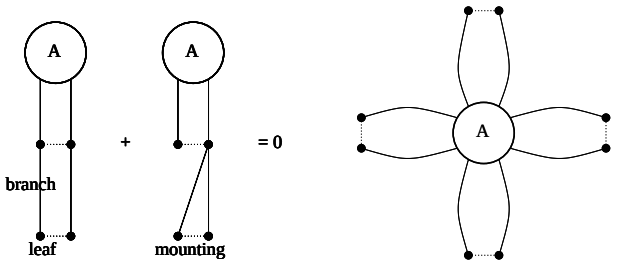

Figure 5: Vanishing contribution to the average of A : the second graph is obtained by mounting a branch to the first graph. The only remaining graphs are simple "flowers".

The graphs which remain at $T = 0$ have $t = 0$, $l = 0$, $k = c$, which means the following properties (i) their subgraph $\mathcal{K}$ has no loop, each of its connected component is a tree, (ii) there is no tadpole on A, (iii) A is linked to each connected component of $\mathcal{K}$ by one unique propagator. This result is easily extended to a non-Gaussian disorder, which possess higher cumulants of the general form (69).

The second part of the argument uses the property of translation invariance in u space of the disorder distribution, on which the first part does not rely. Since each connected

component of $\mathcal{K}$ is linked to A, let us call *root* the point to which it is attached. This provides a natural orientation to the *branches* of the trees from root to *leaves*. If any point of $\mathcal{K}$ possess at least a branch going to the direction of the leaves, the graph obtained by mounting this branch to the companion point (which belongs to the same R before splitting) has the opposite value. One can convince oneself that such graphs can be grouped by mutually cancelling pairs. Thus the only graphs which survive to this mounting operation look like *flowers*, with A at the center and petals R made of two propagators (see Fig. 5).

The generalization including higher cumulants is straightforward, but yields a non-Gaussian theory. The corresponding equivalent action for computing observables is

$$\mathcal{S}_{\rm rf}[u] \;=\; \frac{1}{2T}\sum_a \int_{xy} u_a^x (\overline{G}^{-1})^{xy} u_a^y - \sum_{N\geq 2} \frac{1}{N!T^N} \partial_{1\dots N} S^{(N)}(0\dots 0) \int_x (\sum_a u_a^x)^N \qquad (224)$$

Even if this is not obvious on (224), this action possess statistical tilt symmetry as can be checked thanks to $n \to 0$.

D.2 Corrections to R

The computation of the effective action Γ (1PI) involves corrections to the various cumulants of the disorder. At $T = 0$, the graphs correcting a N replica term ($N^{\rm th}$ cumulant) is made of N connected components, so that there exists a free sum over N replica indices. The power of T in front of such a cumulant has to be $-N$. The graphs correcting R with p R's are made of two connected components ($c = 2$), the power of T is $e - v$ where e is the number of propagators and $v = 2p$. Euler relation yields $e - v = l - c \geq -2$ with equality for $l = 0$. Hence such graphs are made of two trees.

Furthermore, the graphs such that the two points of a splitted R are connected to the same connected component and such that one of them is connected to at least two branches vanish. This can be seen by mounting one of these two branches on the companion point. Hence, if two points of a R belong to the same connected component, then each one is connected to a unique branch. As a corrolary, the two points of an R cannot be connected to each other by a branch (since it would be impossible to connect this R to the rest of the diagram thanks to the argument above).

This considerably reduces the form of the possible corrections to R. These corrections obey in particular

$$\delta R = (\epsilon - 4\zeta)R + \zeta u R' + \sum_{p>0} \left(\frac{d}{du}\right)^{4(p-1)} R^p \qquad (225)$$

where the last term symbolically only means that the $p^{\rm th}$ order term contains $4(p-1)$ derivatives (not that it is a total $4(p-1)$ derivative). We allowed for a field rescaling with exponent ζ. To order R^3 the arguments above allow only for the following corrections

$$\delta R = (\epsilon - 4\zeta)R + \zeta u R' + K(\frac{1}{2}R''^2 - R''R''(0)) + A(R'' - R''(0))R'''^2 + C(R'' - R''(0))^2 R'' \qquad (226)$$

with some constants K, A and C, and valid only for an analytic $R(u)$. In the periodic case ($\zeta = 0$), the fixed point equation is easily solved since there exists to any order in ϵ a fixed

point function of the form

$$R^*(u) = a + bu(1-u) + c\left(u(1-u)\right)^2 \tag{227}$$

where a, b, c can be computed in series of ϵ, once the coefficients of the fixed point equation are known. This is further examined in Sections F.

E variational calculation

Here we sketch the derivation of the scaling function for the confined interface using the replica variational method, extending the explicit solution of Ref. [27] to a non zero mass. We use all notations of Ref. [27] and Ref. [23].

The disorder correlator for the random field problem studied here corresponds to the case $\gamma = 1/2$ and $g = \sigma$ for the parameters of [27]. Applying the variational ansatz for $N = 1$ components yields the function $\tilde{f}(x) = \hat{g}\sqrt{r_f^2 + x}$ which describes the correlations, and $\hat{g} = \sigma\sqrt{2/\pi}$. We have artifially extended the correlator to small scales, so as to obtain a well defined $T = 0$ limit. The large scale results however are independent of the small scale details in the limit of small m. The variational equations reads:

$$(r_f^2 + 2TB(u))^{3/2} = \hat{g} j_D ([\sigma](u) + m^2)^{-\epsilon/2} \tag{228}$$

with $j_D = \int_k (k^2+1)^{-2} = \Gamma(2-d/2)/(4\pi)^(d/2)$ and the equation for the breakpoint (see Ref. [23]) is: $(r_f^2 + 2TB(u_c))^{3/2} = \hat{g} j_D (\Sigma_c + m^2)^{-\epsilon/2}$. This yields the solution:

$$[\sigma](u) + m^2 = A u^{2/\theta} = \Sigma_c (u/u_c)^{2/\theta} \quad u^* < u < u_c \tag{229}$$

$$\sigma(u) = \sigma(0) = A^{\theta/2} \frac{2}{2-\theta} m^{2-\theta} \quad u < u^* \tag{230}$$

and $[\sigma](u) = \Sigma_c$ for $u > u_c$. Here $\theta = \frac{6-\epsilon}{3}$ is the free energy fluctuation exponent and $A = (\frac{\epsilon}{6T}\hat{g}^{2/3} j_D^{-1/3})^{2/\theta}$. This solution allows to compute the small q behaviour of the correlation (for small $m \sim q$) asL

$$\overline{u_q u_{-q}} = \frac{T}{q^2+m^2}\left(1 + \int_{u_c(m^2/\Sigma_c)^{\theta/2}}^{u_c} \frac{du}{u^2} \frac{\Sigma_c (u/u_c)^{2/\theta} - m^2}{q^2 + \Sigma_c (u/u_c)^{2/\theta}} + \sigma(0)\frac{1}{q^2+m^2}\right) \tag{231}$$

which yields the large scale result 143 given in the text. In $D = 0$ one recovers the result of [28]

$$\overline{u^2} = \frac{3}{(4\pi)^{1/3}} m^{-8/3} \sigma^{2/3} \tag{232}$$

F FRG to two loops

F.1 Method with $\mathcal{V}$

The exact RG equation to order U^3 given in C.4 when expanded to two loops yields the following finite temperature RG equation for $\tilde{R}_l(u)$ at large l (the derivation is detailed in [16]):

$$\partial_l \tilde{R}_l = \epsilon \tilde{R}_l + \hat{T}_l \tilde{R}_l'' + K_{l\mu}\left(\frac{1}{2}\tilde{R}_\mu''(u)^2 - \tilde{R}_\mu''(0)\tilde{R}_\mu''(u)\right) \tag{233}$$

$$+K^A(\tilde{R}'' - \tilde{R}''(0))\tilde{R}'''^2 + K^C(\tilde{R}'' - \tilde{R}''(0))^2\tilde{R}'''' \tag{234}$$

$$+K^S\partial_{12}\tilde{S}_l(u,u,0) + \hat{T}_l K^E_{l\mu}\left(\tilde{R}''''_\mu(\tilde{R}''_\mu(u) - \tilde{R}''_\mu(0)) - \tilde{R}''''_\mu(0)\tilde{R}''_\mu\right) + \hat{T}_l K^F_{l\mu}\tilde{R}'''^2_\mu \tag{235}$$

In this formula all terms of order R^2 and higher are retarded, and integrals $\int_0^l d\mu$ are understood. For the R^3 terms we have ommitted the retardation integrals (which involve an additional integral $\int_0^\mu d\nu$) because near the fixed point they can be replaced by a single number. The feedback of the three replica term is through its partial derivatives. This three replica term satisfies its own RG equation given in [16]. The precise values of all coefficients are detailed in [16].

To obtain the large l limit of this equation and thus the fixed point equation we use the fact that $\hat{T}_l \to 0$ and the property (117) which we need only to lowest order. Integrating out the third cumulant RG equation (which reaches the fixed point value (103)) and inserting into (235) yields the FRG fixed point equation (for simplicity in the periodic case):

$$0 = \epsilon\tilde{R}^* + \frac{1}{2}\tilde{R}^{*\prime\prime 2} - \tilde{R}^{*\prime\prime}(0)\tilde{R}^{*\prime\prime} \tag{236}$$

$$+(\frac{1}{2} + \overline{K}^C)(\tilde{R}^{*\prime\prime} - \tilde{R}^{*\prime\prime}(0))\tilde{R}^{*\prime\prime\prime 2} - \frac{1}{2}\tilde{R}^{*\prime\prime\prime}(0^+)^2\tilde{R}^{*\prime\prime} + \overline{K}^C(\tilde{R}^{*\prime\prime} - \tilde{R}^{*\prime\prime}(0))^2\tilde{R}^{*\prime\prime\prime\prime} \tag{237}$$

where only one coefficient, $\overline{K}^C$, is non universal. We have absorbed the exact one loop coefficient K obtained in (91) in R (whose first $\mathcal{O}(\epsilon)$ correction is non universal). In terms of the cutoff function $c(s) = \int_a \hat{c}(a)e^{-as}$ (or its expression as a sum of exponentials) one has:

$$\mathsf{K} = \frac{1}{2(2\pi)^{d/2}}[1 + \epsilon(\frac{1}{2} - \int_a \hat{c}(a)\ln a + \frac{3}{2}\int_{ab}\hat{c}(a)\hat{c}(b)\ln(a+b))]$$

$$\overline{K}^C = -\int_a \hat{c}(a)\ln a + \int_{ab}\hat{c}(a)\hat{c}(b)\ln(a+b) = \int_0^\infty ds\,\frac{c(s)(1-c(s))}{s} \tag{238}$$

to the desired order in ϵ. This equation has a fixed point solution:

$$\tilde{R}^*(0) = \mathsf{K}^{-1}[\epsilon(\frac{1}{2592} - \frac{1}{72}u^2(1-u)^2) + \tag{239}$$

$$\epsilon^2(\frac{1+10\overline{K}^C}{7776} - \frac{\overline{K}^C}{216}u(1-u) - \frac{1+3\overline{K}^C}{108}u^2(1-u)^2)] \tag{240}$$

Note that this solution, for $\overline{K}^C = 0$ (i.e hard cutoff) does not contain a term with $R'(0^+) \neq 0$ ("supercusp")

F.2 method with $\hat{\mathcal{V}}$

The method using the Wick ordered functional can also be used. Using the RG equation to third order in $\hat{U}$ and two loops (189) one finds the equation for the corresponding two replica part of $\hat{U}$:

$$\partial_l \tilde{R}_l = (\epsilon - 4\zeta)\tilde{R}_l + \zeta u \tilde{R}'_l + K\left(\frac{1}{2}\tilde{R}''_l(u)^2 - \tilde{R}''_l(0)\tilde{R}''_l(u)\right) \tag{241}$$

$$+K^A(\tilde{R}'' - \tilde{R}''(0))\tilde{R}'''^2 + \hat{T}_l K_l^F \tilde{R}'''^2_l \tag{242}$$

with:

$$K = -\frac{4}{T^2} I_l^D \Lambda_l^\epsilon \tag{243}$$

$$K_{l\mu}^A = \frac{4}{T^4}(I_{l\mu}^A + I_{l\mu}^{A'}) \tag{244}$$

$$K_{l\mu}^F = -\frac{6}{T^2} I_l^F \Lambda_l^\epsilon \tag{245}$$

where the integrals have been defined in Section C.3. Calculation shows that the constant $K^A = 1/2$ (to lowest order in ϵ) independent of the cutoff function $c(s)$. The coefficient of this term is in agreement with [44, 15]. The analysis of the boundary layer is more intricate in this formulation [16].

References

[1] K.G. Wilson and J. Kogut, Phys. Rep. **12** (1974) 75.

[2] F. Wegner and A. Houghton, Phys. Rev. A **8** (1973) 401.

[3] T.R. Morris, Int. J. Mod. Phys. A9 (1994) 2411

[4] J. Polchinski, Nucl. Phys. B **231** (1984) 269.

[5] T. R. Morris hep-th/9810104.

[6] C. Wetterich, Nucl. Phys. B **352** (1991) 529.

[7] A. Hasenfratz and P. Hasenfratz, Nucl. Phys. B **270** (1986) 685.

[8] J. Berges, N. Tetradis and C. Wetterich hep-ph/0005122.

[9] G.R. Golner and E.K. Riedel, Phys. Rev. Lett. **34** (1975) 856.

[10] J.F. Nicoll, T.S. Chang and H.E. Stanley, Phys. Rev. **A13** (1976) 1251.

[11] T.R. Morris, Phys. Lett. B329 (1994) 241;

[12] D. Zanchi, H. J. Schulz, Z. Phys. B **103** (1997) 339; M. Salmhofer cond-mat/9706188 (*Renormalization* Springer (1998)); C.J. Halboth and W. Metzner, Phys. Rev. B **61** (2000) 4364.

[13] J. Hughes and J. Liu, Nucl. Phys. B **307** (1988) 183.

[14] D.S. Fisher, Phys. Rev. Lett. **56** (1986) 1964. D.S. Fisher, Phys. Rev. B **31** (1985) 7233.

[15] P. Chauve, P. Le Doussal and K. Wiese, cond-mat/0006056 and to be published.

[16] P. Chauve, P. Le Doussal, to be published.

[17] K. B. Efetov and A. I Larkin, Sov. Phys. JETP **45** (1977) 1236

[18] A. Aharony, Y. Imry and S.K. Ma, Phys. Rev. Lett. **37** (1976) 1364, G. Grinstein, Phys. Rev. Lett. **37** (1976) 944, G. Parisi, N. Sourlas PRL **43** (1979) 744. J. L. Cardy, Phys. Lett. **125 B** (1983) 470.

[19] The condition (5,7), if strictly imposed, implies that $c(x)$ is non-analytic. However, a non-analytic cutoff can be obtained by a suitable limit of analytic functions.

[20] One has $a_{2n} = \sum_{p\geq n} \frac{1}{(p-n)!} g_{2p} A_{np} (G_l^{x=0}/2)^{p-n}$ with $A_{np} = \prod_{k=n}^{p-1}(n+2k)/(1+2k)$ and $\tilde{a}_{2n} = \sum_{p\geq n} \frac{1}{(p-n)!} \tilde{g}_{2p} A_{np} (K/2)^{p-n}$ with $K = \Lambda_l^{2-D} G_l^{x=0} = S_D \int_{s>0} (2s)^{-\epsilon/2} c(s)$.

[21] U. Schulz, J. Villain, E. Brézin and H. Orland, J. Stat. Phys. **51** (1988) 1.

[22] L. Balents and D.S. Fisher, Phys. Rev. B **48** (1993) 5959.

[23] T. Giamarchi and P. Le Doussal, Phys. Rev. B **52** (1995) 1242.

[24] P. Le Doussal and C. Monthus, to be published.

[25] L. Balents, Europhys. Lett. **24** 489 (1993).

[26] P. Chauve, T. Giamarchi and P. Le Doussal, cond-mat/0002299.

[27] M. Mézard and G. Parisi, J. Phys. I **1** (1991) 809.

[28] M. Mézard and G. Parisi, J. Phys. I **2** (1992) 2231.

[29] T. Hwa and D.S. Fisher, Phys. Rev. Lett. **72** (1994) 2466; Phys. Rev. B **49** (1994) 3136.

[30] T. Nattermann and S. Scheidl, cond-mat/0003052 (to be published in Adv. Phys.).

[31] T. Emig and M. Kardar, cond-mat/0003031.

[32] D.S. Fisher and D.A. Huse, Fisher Huse. Phys. Rev. B **38** (1988) 386.

[33] E. Orignac, T. Giamarchi and P. Le Doussal, Phys. Rev. Lett. **83** (1999) 2378 and to be published.

[34] For a review of field theoretical methods for this problem see K. Wiese in Domb et Lebowitz vol 19, Academic Press London (2000).

[35] D. Carpentier and P. Le Doussal, Phys. Rev. Lett. **81** 2558 (1998) and cond-mat/9908335, cond-mat/0003281.

[36] Ph. Nozieres and F. Gallet, J. Phys. France **48** (1987) 353; A. Hazareesing, J.-P. Bouchaud, Eur. Phys. J. B **14** (2000) 713.

[37] J. V. Jose et al. Phys. Rev. B **16** (1977) 1217.

[38] N. Tetradis and C. Wetterich, Nucl. Phys. B **422** [FS] (1994) 541.

[39] C. Wetterich, Nucl. Phys. B352 (1991) 529

[40] T.R. Morris, *in* RG96, hep-th/9610012.

[41] T.R. Morris, Phys. Lett. B334 (1994) 355.

[42] T.R. Morris, Nucl. Phys. B495 (1997) 477

[43] T.R. Morris, Nucl. Phys. B458 (1996) 477

[44] H. Bucheli, O. S. Wagner, V. B. Geshkenbein, A. I. Larkin, G. Blatter Phys. Rev. B **57** 1998 7642

[45] P. Chauve, P. Le Doussal in preparation.